AN INTRODUCTION TO MATHEMATICAL PHYSICS

R. A. HOUSTOUN

AN INTRODUCTION TO MATHEMATICAL PHYSICS

AN INTRODUCTION

TO

MATHEMATICAL PHYSICS

BY

R. A. HOUSTOUN, M.A., PH.D., D.SC.

LECTURER ON PHYSICAL OPTICS AND ASSISTANT TO THE PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW

LONGMANS, GREEN, AND CO.

39 PATERNOSTER ROW, LONDON

NEW YORK, BOMBAY, AND CALCUTTA

1912

PREFACE

THIS book is the substance of lectures I have given during the past six years to the Natural Philosophy Class A in the University of Glasgow.

It is intended primarily as a class-book for mathematical students and as an introduction to the advanced treatises dealing with the subjects of the different chapters, but since the analysis is kept as simple as possible, I hope it may be useful for chemists and others who wish to learn the principles of these subjects. It is complementary to the text books in dynamics commonly used by junior honours classes.

A knowledge of the calculus and a good knowledge of elementary dynamics and physics is presupposed on the part of the student.

A large proportion of the examples has been taken from examination papers set at Glasgow by Prof. A. Gray, LL.D., F.R.S., to whom I must also express my indebtedness for many valuable suggestions. The proofs have been read with great care and thoroughness by Dr. John M'Whan of the Mathematical Department.

R. A. HOUSTOUN.

CONTENTS

CHAPTER I.

ATTRACTION.

CHAPTER II.

HYDRODYNAMICS.

CHAPTER III.

FOURIER SERIES AND CONDUCTION OF HEAT.

CHAPTER IV.

WAVE MOTION.

CHAPTER V.

ELECTROMAGNETIC THEORY.

CHAPTER VI.

THERMODYNAMICS.

INTRODUCTION TO MATHEMATICAL PHYSICS.

CHAPTER I.

ATTRACTION.

§ 1. ACCORDING to Newton's law of gravitation every particle of matter attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. Thus if m and m' be the masses of two particles and d the distance between them, F the attraction between them is given by

$$\mathsf{F}=\frac{kmm'}{d^2},$$

where k is the gravitational constant. Newton was at first led to this law by astronomical considerations; he found that it completely explained the motions of the planets. Afterwards, by calculating the force necessary to retain the moon in her orbit, he found that it was this same force that operated between the planets that caused a stone to fall to the earth, and so he was led to postulate the law for all matter. Since Newton's time the law has been repeatedly verified for two bodies on the surface of the earth by such experiments as the Cavendish experiment, and at the same time the value of k has been determined. If F, m, m' and d are measured in dynes, grammes and centimetres, the numerical value of k, according to Poynting, is $6{\cdot}6984 \times 10^{-8}$. Experiments have been made to determine whether the attraction on a crystal depends on the orientation of its axis or whether k varies with the temperature of the bodies, but all such experiments have led to negative results.

Two point charges of electricity act on one another with a force varying as the product of the charges and inversely as the square of the distance between them. Also, if we have two long thin magnets, the poles of which may be considered to be concentrated in points at the ends, there is a force between each pair of poles proportional to the product of the pole strengths and inversely proportional to the square of their distance apart. The attraction between electric charges and between magnetic poles is thus analytically the same as that between gravitating particles. Consequently any result which holds for

gravitational attraction can also be interpreted in terms of electrostatic charges and magnetic poles. The unit quantity of electricity on the electrostatic system and the unit quantity of magnetism on the electromagnetic system are defined so that in the equations analogous to

$$\mathsf{F}=\frac{kmm'}{d^2},$$

k, the constant of proportionality, is unity, when the medium in question is air. Thus, in transferring a result from gravitational attraction to electrostatics, if the medium is air, the constant k must be put equal to unity.*

One fundamental difference there is between gravitational attraction and the action between electric charges and between magnetic poles, namely, as will be explained in Chapter V., that the latter is propagated with a finite velocity from point to point and the medium transmitting it is in a state of stress. If we have a point charge of electricity, the field intensity at a point P, distant r from it, is given by e/r^2. If by any possibility the point charge were suddenly doubled in magnitude, then the field intensity would not double in value at the same instant, but the increase would take a finite interval to travel out to P. But for aught we yet know, in the analogous case of gravitational attraction the intensity would double everywhere instantaneously throughout the whole field.

We shall now calculate the force of attraction, or more shortly the attraction, in some particular cases.

§ 2. Uniform rod at an external point.

Let AB be the rod, P the external point. We suppose that a particle of unit mass is placed at P and that we are required to calculate the attraction of the rod on this particle. The thickness of the rod is supposed to be very small in comparison with its length.

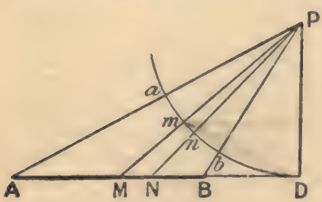

FIG. 1.

Let λ be the linear density of the rod, that is, its mass per unit length. Let PD be drawn from the point perpendicular to the rod and let p be the length of PD. With P as centre and p as radius draw an arc of a circle. Let MN be an element of the rod and let PM and PN meet the arc in m and n.

Then the attraction of MN at P is

$$\frac{k\lambda\mathsf{MN}}{\mathsf{PM}^2}.$$

Also, $\qquad$ area PMN : area $\mathsf{P}mn :: p\,.\,\mathsf{MN} : p\,.\,mn$.

*For the case of other media see p. 142.

But area PMN : area Pmn :: PM^2 : Pm^2, since, of course, the angle MPN is small. Therefore

$$\mathrm{MN} : mn :: \mathrm{PM}^2 : \mathrm{P}m^2 \quad \text{or} \quad \frac{\mathrm{MN}}{\mathrm{PM}^2} = \frac{mn}{\mathrm{P}m^2} = \frac{mn}{p^2}.$$

The attraction of MN at P is thus

$$\frac{k\lambda mn}{p^2}.$$

If we suppose the arc ab uniformly loaded with matter so that its linear density is the same as that of the rod, then its resultant attraction is equal to the resultant attraction of the rod.

The direction of the resultant attraction of the arc must bisect the angle aPb. Let $\angle a\mathrm{P}b = 2\alpha$. Then the attraction due to an element of the arc of length $p\,d\theta$ at an angular distance θ from the direction of the resultant attraction is

$$\frac{k\lambda\, d\theta}{p}.$$

The component of this in the direction of the resultant is

$$\frac{k\lambda\, d\theta \cos\theta}{p}.$$

Hence the resultant attraction of the arc, *i.e.* of the rod, is given by

$$\frac{k\lambda}{p}\int_{-\alpha}^{+\alpha} \cos\theta\, d\theta = \frac{2k\lambda \sin\alpha}{p}.$$

§ 3. **Uniform circular disc at a point on its axis.**

Let a be the radius of the disc and let P be situated a height c above its plane. Let the disc be very thin and let λ be its surface density, *i.e.* the mass of the disc per sq. cm. of surface. Describe with the centre C two adjacent concentric circles, one with radius CA $= r$ and the other with radius CB $= r + dr$. Then the mass of the ring is $2\pi\lambda r\, dr$. Every particle in the ring is at a distance $\sqrt{(c^2 + r^2)}$ from P; also the resultant attraction of the ring is along the axis of the disc. To obtain the component in this direction of the attraction of every particle,

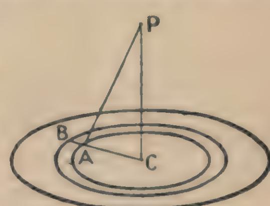

FIG. 2.

we have to multiply that attraction by $\dfrac{\mathrm{CP}}{\mathrm{PA}}$ or $\dfrac{c}{(c^2 + r^2)^{\frac{1}{2}}}$. Hence the resultant attraction of the ring is equal to

$$\frac{2\pi k\lambda r\, dr}{(c^2 + r^2)}\,\frac{c}{(c^2 + r^2)^{\frac{1}{2}}} \quad \text{or} \quad \frac{2\pi k\lambda c r\, dr}{(c^2 + r^2)^{\frac{3}{2}}}.$$

The resultant attraction of the whole disc is

$$2\pi k\lambda c\int_0^a \frac{r\, dr}{(c^2 + r^2)^{\frac{3}{2}}} = 2\pi k\lambda c\left\{\frac{1}{c} - \frac{1}{(c^2 + a^2)^{\frac{1}{2}}}\right\}.$$

If we suppose a to become infinite, we obtain for the attraction of an infinite lamina on an external particle the expression $2\pi k\lambda$, which is independent of the distance of the attracted particle from the lamina.

§ 4. Two thin uniform rods AB and CD have lengths $2a$ and $2c$, and their linear densities are respectively λ and λ'. The midpoint E of AB, and the midpoint F of CD, are a distance b apart, and AB, CD and EF are mutually at right angles to one another. Determine the attraction between the two rods.

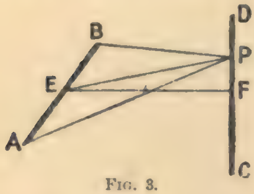

Fig. 3.

Consider an element of CD of length dx at the point P, distant x from F. The mass of this element is $\lambda' dx$, and the attraction exerted on it by the rod AB is, by § 2,

$$\frac{2k\lambda\lambda' dx \sin \mathsf{APE}}{\mathsf{PE}} \quad \text{or} \quad \frac{2k\lambda\lambda' dx}{\sqrt{(b^2+x^2)}} \frac{a}{\sqrt{(a^2+b^2+x^2)}},$$

since AEP is a right angle and $\mathsf{AP}^2 = \mathsf{AE}^2 + \mathsf{EF}^2 + \mathsf{FP}^2$. The direction of the attraction on the element is along PE. By symmetry the resultant attraction of AB on CD must be along EF; hence we need only consider the component in this direction of the attraction on the element. To obtain the component we have to multiply the resultant attraction on dx by cos PEF or $\dfrac{b}{\sqrt{(b^2+x^2)}}$. This gives us

$$\frac{2k\lambda\lambda' ab\, dx}{(b^2+x^2)\sqrt{(a^2+b^2+x^2)}}.$$

The resultant attraction between the rods must therefore be

$$\int_{-c}^{+c} \frac{2k\lambda\lambda' ab\, dx}{(b^2+x^2)\sqrt{(a^2+b^2+x^2)}} \quad \text{or} \quad 4k\lambda\lambda' ab \int_0^c \frac{dx}{(b^2+x^2)\sqrt{(a^2+b^2+x^2)}}.$$

To evaluate the integral, assume $x = b\tan\theta$, so that

$$dx = b\sec^2\theta\, d\theta, \quad b^2+x^2 = b^2\sec^2\theta, \quad \sqrt{(a^2+b^2+x^2)} = \sqrt{(a^2+b^2\sec^2\theta)}.$$

This gives for the resultant attraction

$$4k\lambda\lambda' a \int_0^{\tan^{-1}\frac{c}{b}} \frac{d\theta}{\sqrt{(a^2+b^2\sec^2\theta)}} = 4k\lambda\lambda' \int_0^{\tan^{-1}\frac{c}{b}} \frac{a\, d(\sin\theta)}{\sqrt{(a^2+b^2-a^2\sin^2\theta)}}$$

$$= 4k\lambda\lambda' \left\{ \sin^{-1}\left(\frac{a\sin\theta}{\sqrt{(a^2+b^2)}} \right) \right\}_0^{\tan^{-1}\frac{c}{b}}$$

$$= 4k\lambda\lambda' \sin^{-1} \frac{ac}{\sqrt{\{(a^2+b^2)(b^2+c^2)\}}}.$$

§ 5. **Homogeneous spherical shell at an external point.**

A thin spherical shell is a solid bounded by two concentric spheres of almost equal radius.

Let C be the centre of the shell, a its radius and λ its mass per unit area of external surface. It is required to find the force with which the shell would attract a particle of unit mass situated at P. Let $CP = c$.

Consider the ring cut off from the shell by the rotation about CP of the angles PCM and PCN, which are respectively equal to θ and $\theta + d\theta$. Its radius is $a \sin\theta$; $NM = a\,d\theta$. Hence its mass is $2\pi a^2\lambda \sin\theta d\theta$.

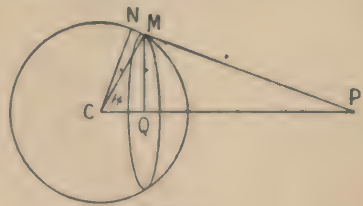

FIG. 4.

From considerations of symmetry, the resultant attraction of the ring must be along PC, but the attraction of each individual particle in it is in the straight line towards that particle. The particles in the ring are all at the same distance from P, namely PM. In order to obtain the component of their attractions in the direction CP, we multiply by cos CPM or PQ/PM. The resultant attraction of the ring is therefore

$$\frac{2\pi a^2 k\lambda \sin\theta d\theta}{PM^2}\,\frac{PQ}{PM}.$$

Let us now change the independent variable to y, y being equal to PM. We have

$$y^2 = PM^2 = c^2 + a^2 - 2ac\cos\theta.$$

Therefore $$y\,dy = ac\sin\theta\,d\theta.$$

Also $$PQ = c - a\cos\theta = \frac{1}{2c}(2c^2 - 2ac\cos\theta) = \frac{1}{2c}(y^2 + c^2 - a^2).$$

On substituting for θ, the resultant attraction of the ring becomes

$$\frac{\pi a k\lambda}{c^2}\left(1 + \frac{c^2 - a^2}{y^2}\right)dy.$$

To obtain the attraction of the whole shell, we have to integrate this between the limits $y = c - a$ and $y = c + a$. The result is

$$\frac{\pi a k\lambda}{c^2}\int_{c-a}^{c+a}\left(1 + \frac{c^2 - a^2}{y^2}\right)dy = \frac{\pi a k\lambda}{c^2}\left(y - \frac{c^2 - a^2}{y}\right)_{c-a}^{c+a}$$

$$= \frac{4\pi a^2 k\lambda}{c^2} = \frac{k \text{ mass of shell}}{c^2}.$$

Therefore the shell attracts a particle at an external point as if its whole mass were collected at its centre.

§ 6. **Homogeneous spherical shell at an internal point.**

We proceed as in § 5, but the limits of y are $a-c$ and $a+c$. Hence the attraction of the whole shell is

$$\frac{\pi ak\lambda}{c^2}\int_{a-c}^{a+c}\left(1+\frac{c^2-a^2}{y^2}\right)dy=\frac{\pi ak\lambda}{c^2}\left(y-\frac{c^2-a^2}{y}\right)_{a-c}^{a+c}=0.$$

The resultant attraction of the shell at an internal point is therefore zero.

§ 7. **Homogeneous spherical shell at an internal point. Otherwise.**

Before entering on this alternative proof it is necessary to define a solid angle. Let S be a surface which is not necessarily plane. In order to measure the solid angle subtended at the point P by the surface S, we draw a sphere of unit radius with its centre at P. Let straight lines be drawn from P to every point in the circumference of S. Then these straight lines will be the generators of a cone, and this cone will intercept a certain area on the surface of the sphere. The solid angle subtended by the surface S at P is numerically equal to the area intercepted on the sphere of unit radius. Thus, if the surface S be closed and P be an internal point, the solid angle is 4π; if P be an

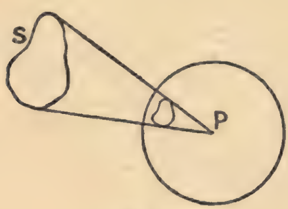

Fig. 5.

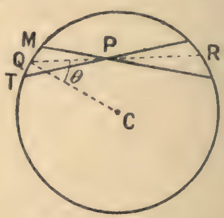

Fig. 6.

external point, it is zero, for the tangents drawn from P will touch S in points lying on a curve which divides S into two parts S_1, S_2, and the area subtended by the one part S_1 of the surface on the sphere of unit radius is numerically equal to and of opposite sign from the area subtended by the other part S_2.

Let fig. 6 represent the shell and let P be the internal point. With P as vertex draw a cone of small vertical solid angle $d\omega$. Let PQ and PR, the distances of P from the shell measured along the axis of the cone, be r_1 and r_2; $d\omega$ is so small that it is not necessary to specify exactly what is meant by the axis of the cone. Let dS_1 and dS_2 be respectively the areas intercepted by the cone on the surface of the shell at Q and R.

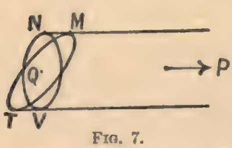

Fig. 7.

Now consider fig. 7. In it MT represents the element of surface dS_1. The vertex of the cone is so far away that in the neighbourhood of dS_1 the cone may be regarded as a cylinder. With P as centre and PQ

as radius describe a sphere, and let NV be the portion of the area of this sphere intercepted by the cone. Then $NV = r_1^2 d\omega$. But from fig. 6, if C be the centre of the sphere and θ be the angle RQC, it is obvious that θ is the angle between the normals to NV and MT. Thus

$$NV = MT \cos\theta, \quad r_1^2 d\omega = dS_1 \cos\theta \quad \text{or} \quad dS_1 = \frac{r_1^2 d\omega}{\cos\theta}.$$

Similarly,
$$dS_2 = \frac{r_2^2 d\omega}{\cos\theta}.$$

The mass of the portion of the shell intercepted by the cone at Q is $\lambda r_1^2 d\omega/\cos\theta$. The attraction it exerts at P is in the direction PQ and of amount $\dfrac{k\lambda r_1^2 d\omega/\cos\theta}{r_1^2}$, which is equal to $k\lambda d\omega/\cos\theta$. Similarly the attraction exerted by the portion intercepted at R is in the opposite direction and of the same amount. The resultant attraction of the ends of the cones is therefore zero. But the whole shell may be divided up in this way into an infinite number of cones. Hence the resultant attraction of the whole shell is zero.

§ 8. **Elliptic homœoid. Internal point.**

Suppose that in the case of the previous example all lengths in the direction of the x-axis are increased a times, all lengths in the direction of the y-axis b times and all lengths in the direction of the z-axis c times. Then the inner and outer surfaces of the shell will become similar, concentric and similarly situated ellipsoids, as shown in fig. 8. The cone will still be a cone and the masses intercepted by its ends remain unaltered. The ratio of QP to PR also remains the same. Hence the attractions of the ends at Q and R still balance. A solid bounded by two similar, concentric, similarly situated ellipsoids of nearly equal magnitude is called a thin elliptic homœoid. By dividing up its surface by an infinite number of cones it can thus be shown that the attraction it exerts at an internal point is zero.

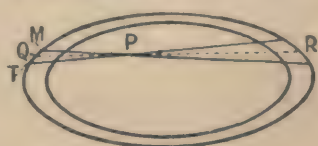

FIG. 8.

The converse of this theorem is true and is of importance in electrostatics. The electric intensity at any point inside a charged conductor is zero. The charge is situated on the outside. Hence, if the surface of the conductor be an ellipsoid, the density of the charge is given by the above theorem, that is, it is proportional to the thickness at the point of the thin elliptic homœoid which has the ellipsoid as one of its surfaces.

§ 9. **Attraction at its pole of a homogeneous solid oblate spheroid of small eccentricity.**

Let $2b$ be the length of the minor axis and $2a$ that of the major axis of the generating ellipse. The spheroid may be supposed made up of

a concentric sphere, the radius of which is b, and an exterior shell. The attractions of these portions will be calculated separately.

Take the axis of revolution as the axis of y and the centre of the sphere as the origin of coordinates, and let a plane be drawn perpendicular to the axis of revolution at a distance y from the origin to cut the sphere and spheroid in two concentric circles, as in fig. 9. The

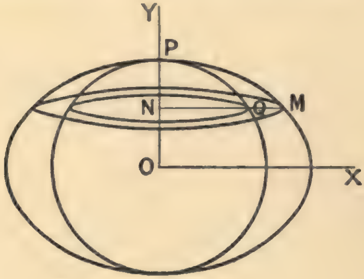

FIG. 9.

area of the inner circle is given by $\pi \mathrm{NQ}^2 = \pi(b^2 - y^2)$. The area of the outer circle is given by $\pi \mathrm{NM}^2$, which equals $\pi a^2\left(1 - \frac{y^2}{b^2}\right)$ since M is a point on the spheroid. The area of the ring is therefore

$$\pi\left(a^2 - \frac{a^2y^2}{b^2} - b^2 + y^2\right) = \pi(a^2 - b^2)\left(1 - \frac{y^2}{b^2}\right).$$

Let another plane be drawn parallel to the first at a distance dy from it; then the mass of the element of the shell comprised between the two planes is

$$\pi\rho(a^2 - b^2)\left(1 - \frac{y^2}{b^2}\right) dy,$$

where ρ is the density of the spheroid.

The distance of every particle in this element from P is given by PQ, since QM is small. To find the component of the attraction in the direction NP we multiply by PN/PQ.

$$\frac{\mathrm{PN}}{\mathrm{PQ}^3} = \frac{(b-y)}{\{(b-y)^2 + x^2\}^{\frac{3}{2}}} = \frac{(b-y)}{\{(b-y)^2 + b^2 - y^2\}^{\frac{3}{2}}} = \frac{1}{(2b)^{\frac{3}{2}}(b-y)^{\frac{1}{2}}}.$$

The attraction of the whole shell at P is therefore

$$\pi k\rho(a^2 - b^2)\int_{-b}^{+b} \frac{\left(1 - \frac{y^2}{b^2}\right) dy}{(2b)^{\frac{3}{2}}\sqrt{b-y}}.$$

In order to integrate this put $b-y=z$. Then the integral becomes

$$\frac{\pi k\rho(a^2-b^2)}{2^{\frac{3}{2}}b^{\frac{1}{2}}}\int_0^{2b}(2bz^{\frac{1}{2}}-z^{\frac{3}{2}})\,dz=\frac{\pi k\rho(a^2-b^2)}{2^{\frac{3}{2}}b^{\frac{1}{2}}}\,\frac{2^{\frac{9}{2}}b^{\frac{5}{2}}}{15}=\frac{8\pi k\rho(a^2-b^2)}{15b}.$$

If we suppose $a=b(1+\epsilon)$, where ϵ is small, we have $a^2-b^2=2b^2\epsilon$. Hence the resultant attraction of the shell is

$$\frac{16\pi k\rho\epsilon b}{15}.$$

By § 5 the attraction of the sphere is $\frac{4}{3}\pi k\rho b$; therefore the attraction of the spheroid is

$$\tfrac{4}{3}\pi k\rho b(1+\tfrac{4}{5}\epsilon).$$

EXAMPLES.

1. If a particle be attracted by three uniform rods, joined together at their ends so as to form a triangle, it will be in equilibrium if placed at the centre of the circle inscribed in the triangle.

2. Prove that the resultant attraction of a very long rectangular plate on a particle of unit mass in its plane, in line with one end of the plate, and at distances a, a' from its long edges, is in a line inclined at 45° to these edges, and is of amount $\sqrt{2}k\sigma\log(a'/a)$, where σ is the uniform surface density of the plate.

3. Show that the resultant attraction of a uniform right circular cylinder on a particle situated on its axis outside the cylinder at a distance c from its end is

$$2\pi k\rho[h-\sqrt{\{(c+h)^2+b^2\}}+\sqrt{\{c^2+b^2\}}],$$

where h, b and ρ are respectively the height, radius and density of the cylinder.

4. Show that the resultant attraction of a uniform right circular cone on a particle at its vertex is

$$2\pi k\rho(1-\cos\alpha)h,$$

where h, α, ρ are respectively the height, semi-vertical angle and density of the cone.

5. Find the attraction of a segment of a paraboloid of revolution, bounded by a plane perpendicular to its axis, on a particle at the focus.

Result. $4\pi k\rho a\log\dfrac{x+a}{a}$, where x is the distance of the bounding plane from the vertex.

6. Find an exact expression for the change in g produced by descending a depth h below the earth's surface, on the supposition that the density of the whole spherical surface stratum is the same, σ say, and derive an approximate expression for use at a moderate depth.

In the Harton Colliery experiment $h=1260$ feet; taking the surface density as 2·5 and the mean density as 5·6, find the number of beats at the bottom of the mine made in 24 hours by a pendulum which beats seconds at the surface. (Cf. Gray's *Physics*, p. 526.)

7. Suppose that there is a hollow tube through the earth along one of its diameters and that a particle is dropped down it. How long will it take to reach the other side? (42 minutes.)

§ 10. **Theorem of surface integral of normal force.**

Let a closed surface be drawn in a region of space containing gravitating matter. Let F be the force on a unit particle at an element dS of this surface, measured positive outwards, and let θ be the angle which F makes with the outward drawn normal to dS. Then $\int \text{F} \cos\theta \, d\text{S}$ is the surface integral of normal force. The theorem to be proved may be enunciated as follows.

The surface integral of normal force taken over a closed surface in a field of force due to matter attracting according to the inverse square of the distance is equal to -4π times the quantity of matter within the surface, multiplied by the gravitation constant.

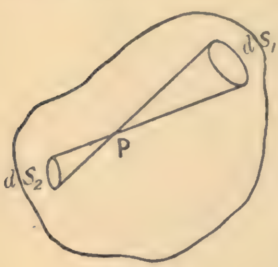

FIG. 10.

First of all suppose that there is a single particle of mass m in the field and that it is inside the surface at P. With P as vertex draw a cone of solid angle $d\omega$ to intercept elements of the surface $d\text{S}_1$, $d\text{S}_2$. Let $d\omega$ be small, let $d\text{S}_1$, $d\text{S}_2$ be respectively distant r_1 and r_2 from P and let the axis of the cone make angles θ_1 and θ_2 respectively with the normals to $d\text{S}_1$ and $d\text{S}_2$. The vertical angle of the cone is taken so small that all its generating lines may be considered parallel. Then, as in § 7, $d\text{S}_1 \cos\theta_1 = r_1^2\, d\omega$, $d\text{S}_2 \cos\theta_2 = r_2^2\, d\omega$. The normal force at $d\text{S}_1$ is $-\dfrac{km}{r_1^2}\cos\theta_1$, and at $d\text{S}_2$ is $-\dfrac{km}{r_2^2}\cos\theta_2$. Multiplying the normal force at each end by the element of area there, we obtain

$$-\frac{km}{r_1^2}\cos\theta_1 \frac{r_1^2\, d\omega}{\cos\theta_1} - \frac{km}{r_2^2}\cos\theta_2 \frac{r_2^2\, d\omega}{\cos\theta_2} = -2km\, d\omega.$$

On dividing up the surface into elements by an infinite number of cones with their vertices at P, it is clear that the sum of the effects of the different elements, the surface integral of normal force, is

$$-4\pi km.$$

If the particle is outside the surface, the cone cuts the latter in two elements which are both on the same side of the vertex. The direction of F is the same on each element, but the direction of the normal is different; hence the products have opposite signs and the two ends cancel one another. For an external point the surface integral of normal force is therefore zero.

Suppose now that the surface has a fold in it so that the cone cuts it more than twice (cf. fig. 11). Since the surface is closed the cone must cut it an even number of times. If the point is an internal one, it is obvious that the effects of the successive elements thus formed will annul one another except in the case of two elements, while if the

point is an external one, the annulment is complete. Thus the theorem still holds if the cone cuts the surface more than twice.

Suppose that instead of one we have several internal particles of masses m_1, m_2, ... m_n, then the theorem will hold for each of them separately. Therefore

$$\int(F_1\cos\theta_1 + F_2\cos\theta_2 \dots F_n\cos\theta_n)dS = -4\pi k(m_1 + m_2 \dots + m_n),$$

F_1, F_2, ... F_n being the forces at dS due respectively to m_1, m_2, ... m_n and θ_1, θ_2, ... θ_n being the angles which the normal to dS makes respectively with the directions of F_1, F_2, ... F_n. For

$$F_1\cos\theta_1 + F_2\cos\theta_2 \dots + F_n\cos\theta_n$$

we may write $F\cos\theta$, where F is the resultant of F_1, F_2, ... F_n and θ the angle which it makes with the direction of the normal to dS, and for $m_1 + m_2 \dots + m_n$ we may write M the total mass inside the surface.

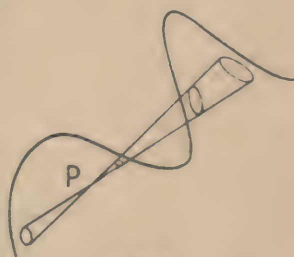

FIG. 11.

Then
$$\int F\cos\theta\, dS = -4\pi k M,$$
and the theorem is proved.

We shall now make some applications of this theorem.

§ 11. Solid sphere, the density of which is a function of the distance from the centre.

If the sphere be divided up into a system of concentric shells of small thickness, the density of each shell is constant. This includes as a particular case the homogeneous sphere. Let M be the total mass of the sphere and a its radius. It is required to determine the attraction at an external point P distant r from the centre.

Through P draw a sphere of radius r concentric with the given sphere and let F be the force on a unit particle at P, measured positive outwards. Then, from considerations of symmetry, F has the same magnitude at all points on the surface of the sphere of radius r, and its direction is everywhere normal to the surface of this sphere. The surface integral of normal force taken over the sphere of radius r is therefore $4\pi r^2 F$. The quantity of matter inside this sphere is M. Hence

$$4\pi r^2 F = -4\pi k M \quad \text{and} \quad F = -\frac{kM}{r^2}.$$

Take now the case of a homogeneous spherical shell, and let P be distant r from the centre of the shell, r being not greater than the radius of the inner surface of the shell. Let F be the force, if any, exerted by the shell on a unit particle at P. Through P draw a sphere

of radius r concentric with the shell. Then, from considerations of symmetry, F must be normal to this sphere, and have the same value everywhere on its surface. The surface integral of normal force taken over the sphere of radius r is therefore $4\pi r^2\mathsf{F}$. But the quantity of matter inside the sphere of radius r is zero. Therefore $\mathsf{F}=0$; that is, the attraction at any point inside the shell is zero.

Let us now return to the solid sphere of radius a, the density of which is a function of the distance from the centre, and let P be an internal point distant r from the centre of the sphere. In order to determine the attraction of the sphere at P draw through P a sphere of radius r, concentric with the given sphere. This sphere divides the given sphere into two portions, for the inner of which P is an external point and for the outer of which P is an internal point. The matter contained between the spheres of radius r and a consequently exerts no action on a particle at P, and the matter contained inside the sphere of radius r acts at P as if it were all concentrated at the centre.

§ 12. **Infinitely long right circular cylinder, the density being a function of the distance from the axis.**

This of course includes the case of the cylinder having a cylindrical hollow core.

By symmetry the attraction is the same at all points on a cylinder coaxal with the given cylinder and is directed normally inwards. Let such a cylinder of radius r be cut by two planes perpendicular to the axis at unit distance apart and let us take the surface integral of normal force over the surface of the cylinder of unit length thus enclosed. On the ends of this cylinder, F is tangential; consequently the ends contribute no part to the surface integral. The area of the convex surface is $2\pi r$. Hence, on applying the theorem,

$$2\pi r\mathsf{F} = -4\pi k\mathsf{M} \quad \text{and} \quad \mathsf{F} = -\frac{2k\mathsf{M}}{r}.$$

M is the mass per unit length of the given cylinder included within the coaxal cylinder of radius r; r can, of course, be either greater or less than the radius of the given cylinder.

§ 13. **Uniform lamina bounded by two parallel planes and extending to an infinite distance in all directions.**

By symmetry the attraction will be normal to the lamina, and its magnitude will be the same at all points equidistant from it, whether on the same side or on opposite sides of it.

Consider a right circular cylinder, the ends of which are parallel to the surface of the lamina and at equal distances on opposite sides of it. Let A be the area of the ends and F the outward force exerted by the lamina on a unit particle situated anywhere on either end. Let M be the mass of the lamina per unit area of surface. The total mass contained inside the cylinder is thus AM. The attraction on the convex surface of the cylinder is tangential to the latter; hence the

convex surface contributes nothing to the surface integral of normal force. Then

$$2AF = -4\pi kAM \quad \text{or} \quad F = -2\pi kM.$$

Thus F is independent of the distance from the lamina.

§ 14. **Potential.**

Let m be the mass of a particle situated at Q which attracts according to the inverse square law and let another particle of unit mass move along a curve from A to B in the field of the first particle. It is required to find the work done during the displacement by the attraction of m.

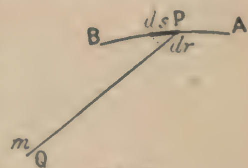

FIG. 12.

Take any element of length ds situated at point P on AB and let $QP = r$. The attraction at P is km/r^2 and acts along PQ. Take the component of ds along PQ and let it be dr. Then the work done in moving the particle of unit mass along ds is $-km\,dr/r^2$. But the whole displacement AB can be divided up into an infinite number of such elements. Hence the work done in the whole displacement is

$$\int_{r=AQ}^{r=BQ} -\frac{km\,dr}{r^2},$$

the negative sign being taken since dr is negative when the work is positive. On integration this becomes

$$km\left(\frac{1}{BQ} - \frac{1}{AQ}\right).$$

It is obvious that this result holds whether the curve AB lies in one plane or not. It is also obvious that if the particle of unit mass is carried round a closed curve the total work done on it is zero.

Displace the point A to infinity and let B coincide with P. Then the work done in bringing the particle of unit mass from infinity to P is equal to km/r. This quantity is defined to be the potential at P.

Suppose that instead of one particle of mass m we have a system of particles of masses $m_1, m_2, \ldots m_n$ distant respectively $r_1, r_2, \ldots r_n$ from the point P. Then, since the work done by the resultant attraction of the system in bringing the particle of unit mass from infinity is equal to the sum of the work done by the attractions of the different particles, we have in this case for the potential at P the expression

$$\frac{km_1}{r_1} + \frac{km_2}{r_2} \ldots + \frac{km_n}{r_n} = \Sigma\frac{km}{r}.$$

So far the definition has been confined to a system of discrete particles; it may also be extended to the case of a continuous distribution of matter. For let ρ be the density at x', y', z' and let the

coordinates of P be x, y, z. Then the mass of the element at x', y', z' is $\rho\, dx'\, dy'\, dz'$ and its potential at P is

$$\frac{\rho\, dx'\, dy'\, dz'}{r},$$

where $r = \sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}$. Consequently the potential of the whole distribution at P is

$$\iiint \frac{\rho\, dx'\, dy'\, dz'}{r},$$

ρ being a function of x', y', z' and the integration being taken throughout all space where there is matter.

We shall denote the potential at x, y, z by V.

Let X, Y, Z be the components of the resultant attraction of the system on the particle of unit mass at the point P(x, y, z), X, Y, Z being taken positive in the positive directions of x, y, z. Let the particle of unit mass be displaced through any element of distance ds, the components of which are dx, dy, dz. Then, from the definition,

$$dV = F'\, ds,$$

where F′ is the component of the attraction of the system in the direction of ds. Hence

$$F' = \frac{\partial V}{\partial s},$$

i.e. the attraction in any direction is equal to the rate of increase of potential in that direction.

Since V is a function of the coordinates, we have

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz.$$

But since the work done by a force is equal to the work done by the components,

$$dV = F'\, ds = X\, dx + Y\, dy + Z\, dz.$$

This equation and the preceding one hold no matter what the values of dx, dy, dz are. Hence

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}.$$

§ 15. **Lines of force and equipotential surfaces.**

If we start from any point and move always in the direction of the resultant force of attraction, we trace out a *line of force*. A line of force may be defined as a curve to which the resultant force is everywhere tangential.

The potential at a point P is a function of the coordinates of that point. We may express this fact by the following equation,

$$V = f(x, y, z).$$

Now $f(x, y, z) = c$, where c is a constant, is the equation to a surface. We see at once that everywhere on this surface the potential has the same value. Such surfaces are called equipotential surfaces.

The direction cosines of the normal to the surface at the point x, y, z are proportional to

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \text{or to} \quad \frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial z},$$

i.e. to X, Y, Z, where X, Y, Z are the components of the attraction at x, y, z. Hence the resultant attraction at x, y, z is at right angles to the surface $f(x, y, z) = c$; the lines of force and the equipotential surfaces cut everywhere orthogonally.

§ 16. Consider two consecutive equipotential surfaces. Let the potential over one of them have the value V and over the other the value $V + \delta V$. Then δV is the work done on unit particle in bringing it from any point on the first surface to any point on the second. Let δn be the distance between the surfaces, measured along a line of force. The average force along this line of force between the two surfaces is

$$\frac{\delta V}{\delta n}.$$

This varies inversely as δn.

By means of the equipotential surfaces, therefore, we can represent the force throughout the whole field in magnitude as well as in direction. For, if we draw the surfaces

$$V = c, \quad V = 2c, \quad V = 3c, \text{ etc.},$$

increasing the constant always by the same amount so as to fill the whole field with surfaces, the work done in taking unit particle from any surface to the one next it is always the same. The direction of the force is given by a curve through the point intersecting the surfaces at right angles and its magnitude is proportional, or if c be chosen suitably, equal to the number of surfaces intersected by this curve in unit length.

§ 17. **Tubes of force.**

Upon an equipotential surface let a small closed curve be drawn. The lines of force which pass through this curve mark out a tubular surface, which is called a tube of force. Take a portion of such a tube bounded by two normal sections, which of course will be elements of equipotential surfaces, and apply to it the surface integral of normal force theorem. Let the areas of the ends be δS_1, δS_2. The normal force on the side is zero because the force is there tangential. The force at the ends is the resultant force there; denote it by F_1, F_2. If there is no matter within the portion of tube contained,

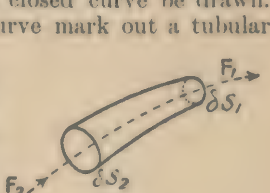

Fig. 13.

$$F_1 \delta S_1 - F_2 \delta S_2 = 0 \quad \text{or} \quad F_1 \delta S_1 = F_2 \delta S_2,$$

that is, the force varies inversely as the cross section of the tube.

Let us suppose the whole field is filled with such tubes and that for each tube the product

$$F\,\delta S = c,$$

where c is small and constant. Then it is easy to see that these tubes represent the field intensity both in magnitude and direction.

The surface integral of normal force over any closed surface is then equal to c times the excess of the number of tubes which cross the surface from within over the number of those which cross it from without. As it also equals $-4\pi k M$, M being the quantity of matter in the enclosed region, one tube starts from every $\frac{c}{4\pi k}$ units of mass. The end of each tube is thus always associated with the same definite quantity of matter.

§ 18. **Potential due to a homogeneous sphere.**

Let a be the radius of the sphere and M its mass. Let r measure the distance of any point P from its centre.

Then, if P be an external point, the attraction at P is kM/r^2 and the potential of P $\frac{kq}{r^2}$

$$\int_{\infty}^{r} -\frac{kM}{r^2}\,dr = \frac{kM}{r}.$$

If P be an internal point the attraction at P is kMr/a^3. The work done in bringing the particle of unit mass from infinity to the surface of the sphere is kM/a. The work done in bringing it from the surface to P is given by

$$\int_{a}^{r} -\frac{kMr}{a^3}\,dr = \frac{kM}{2a^3}(a^2 - r^2).$$

The potential at P is thus

$$kM\left(\frac{1}{a} + \frac{a^2 - r^2}{2a^3}\right) \quad \text{or} \quad kM\left(\frac{3}{2a} - \frac{r^2}{2a^3}\right).$$

Hence the equipotential surfaces, both inside and outside the given sphere, are spheres concentric with the latter.

In the case of a thin uniform spherical shell of mass m and radius a, the work done in bringing the particle of unit mass from infinity to the surface of the shell is km/a. Suppose that the particle is taken through the surface of the shell; the force acting on it is finite, and this part of the path is extremely short. Hence the work done on it can be neglected. There is no force inside the shell. The potential there is thus everywhere km/a.

The potential inside a thick homogeneous spherical shell of mass m bounded by spheres of radii a and b, b being less than a, is given by

$$\frac{3}{2}km\,\frac{a+b}{a^2+ab+b^2}.$$

§ 19. **Potential due to infinitely long cylinder.**

Let the cylinder be homogeneous, right circular, of radius a, and let M be its mass per unit length. Let r be the distance of any point P from the axis of the cylinder.

Then, from § 12, the attraction at an external point is

$$\frac{2k\mathrm{M}}{r}.$$

The work done in bringing a particle of unit mass from infinity to P is

$$\int_{\infty}^{r} -\frac{2k\mathrm{M}}{r}\,dr = 2k\mathrm{M}(\log\infty - \log r),$$

and is hence infinitely great. In this case we measure the potential from the axis as a reference point. The attraction at any internal point P is $2k\mathrm{M}r/a^2$ and the potential is thus:

$$\int_{0}^{r} -\frac{2k\mathrm{M}r}{a^2}\,dr = -\frac{k\mathrm{M}r^2}{a^2}.$$

On the surface of the cylinder it is $-k\mathrm{M}$, and if P be an external point, the work done against the forces of the field in taking the particle from the surface to P is

$$2k\mathrm{M}(\log r - \log a).$$

Hence the potential there is

$$-k\mathrm{M} + 2k\mathrm{M}\log\frac{a}{r}.$$

By this time it will be evident that potential is not merely an aid to studying the energy changes of a certain "particle of unit mass," but is an extremely useful way of obtaining an insight into the distribution of forces in a field. Also, since it is the forces we are concerned with, it is immaterial from what point potential is measured. Changing the reference point merely adds a constant which disappears in the differentiation.

EXAMPLES.

1. Show that a tube of force is refracted when it passes obliquely through a thin layer of matter.

2. Show that two uniform spheres attract each other as if their masses were collected at their centres.

3. A sphere of radius a, mass M and density varying directly as the distance from the centre is built up of matter brought from an infinite distance; show that the work W done throughout the process, by the attraction of the matter which has already arrived on that which is brought up later, is given by $\mathrm{W} = \frac{4}{7}k\mathrm{M}^2/a$, where k is the constant of gravitation.

Prove that this value of the work would not be altered by supposing the matter originally uniformly distributed through infinite space.

Show that if the matter is now redistributed so as to form a sphere of the same radius but of uniform density, work would be done by the mutual attractions of the parts to the extent of $\frac{1}{20}$W.

4. Show that the equipotential surfaces and lines of force of a uniform rod are ellipsoids and hyperbolas having the ends of the rod as foci.

5. A slab is bounded by two parallel infinite planes and its density is a function of the distance from one of these planes. Find the attraction at an internal point, and show how the potential varies in passing through the slab.

6. Supposing a solid homogeneous sphere of mass M and radius a to be held together only by the mutual attractions of its particles, find the force required to separate it into two hemispheres. ($\frac{3}{16}k\mathrm{M}^2/a^2$.)

§ 20. **Gauss's theorem of average potential over a spherical surface.**

The mean value of V over the whole of a spherical surface is equal to the value of V at the centre, provided that none of the attracting matter lies within the surface.

Let m be one of the attracting particles, and let $d\mathrm{S}$ be an element of the spherical surface. Let r be the distance of $d\mathrm{S}$ from m. Then the potential at $d\mathrm{S}$ due to m is m/r. The average value over the sphere of the potential due to m is

$$\frac{\int \frac{m}{r}\,d\mathrm{S}}{\int d\mathrm{S}}.$$

But $\int \frac{m}{r}\,d\mathrm{S}$ may be regarded as the potential which would be produced at the point where m is situated by a thin spherical shell whose mass per unit area is m and which coincides with the spherical surface. We know that this is the same as if the whole mass of the shell were concentrated at its centre.

Let a be the radius of the sphere, d the distance of its centre from the particle. Then

$$\frac{\int \frac{m}{r}\,d\mathrm{S}}{\int d\mathrm{S}} = \frac{\frac{4\pi a^2 m}{d}}{4\pi a^2} = \frac{m}{d},$$

that is, the potential at the centre of the sphere due to the particle.

The theorem thus holds true for a single particle, and as the potential due to a system is equal to the sum of the potentials due to its parts, it must hold for the potential due to the whole external distribution.

It follows from this theorem that V cannot have a maximum or a minimum at a point in empty space. For with such a point as centre it is possible to draw a small sphere containing no matter, and the average value of V over this sphere is equal to the value at the centre. Hence V at such a point cannot be a maximum or minimum.

Again, if V has a constant value V_0 in any region of the field, it must have the same value at every point of the field that can be reached from this region without passing through attracting matter. For if not, let P be a point just inside the region, and let a sphere be drawn with P as centre passing out of the region but not containing any attracting matter. Then the radius of the sphere can be taken so small, that on the part of the surface outside the region V has a value either greater or less than V_0. The average value of V over the surface must therefore be correspondingly greater or less than V_0. But the value at the centre is V_0. Hence the value on the part of the surface outside the region cannot be greater or less than V_0, that is, it must equal V_0. Similarly the region can be extended by drawing other spheres until we come up against attracting matter.

§ 21. **Gauss's theorem.**

Let X, Y, Z be a vector, a continuous function of the coordinates. Let any closed surface be taken. Then if dS is an element of the surface and l, m, n the direction cosines of the outward drawn normal to dS,

$$\iiint\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)dx\,dy\,dz=\iint(lX+mY+nZ)\,dS,$$

the surface integral being taken over the whole surface and the volume integral throughout the region bounded by the surface.

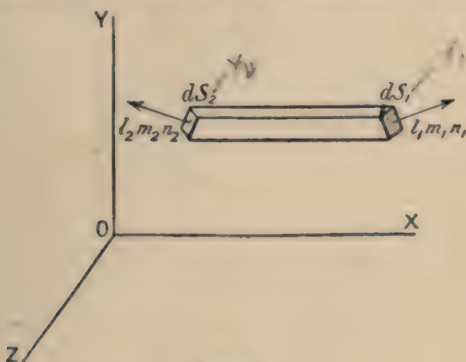

FIG. 14.

Consider $\iiint \frac{\partial X}{\partial x}\,dx\,dy\,dz$. Divide the space bounded by the surface into elementary strips by planes parallel to xy and xz. Let one of these strips, of cross-sectional area $dy\,dz$, be represented in the diagram, and let it intercept areas dS_1, dS_2 on the surface. Let X_1, X_2, $l_1 m_1 n_1$, $l_2 m_2 n_2$ be the values of X and l, m, n at dS_1 and dS_2 respectively.

Integrating along the strip, we obtain

$$\iiint\frac{\partial X}{\partial x}\,dx\,dy\,dz=\iint(X_1-X_2)\,dy\,dz.$$

But $dy\,dz = l_1\,dS_1 = -l_2\,dS_2$, due regard being paid to the sign of l_2.

Therefore $$(X_1 - X_2)\,dy\,dz = l_1 X_1\,dS_1 + l_2 X_2\,dS_2.$$

Integrating through the other strips and adding, we thus obtain

$$\iiint \frac{\partial X}{\partial x}\,dx\,dy\,dz = \iint lX\,dS.$$

Similarly,

$$\iiint \frac{\partial Y}{\partial y}\,dx\,dy\,dz = \iint mY\,dS, \qquad \iiint \frac{\partial Z}{\partial z}\,dx\,dy\,dz = \iint nZ\,dS,$$

and the theorem follows.

§ 22. **Divergence of a vector.**

The expression $$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$$

is said to be the divergence of the vector X, Y, Z, and is written

$$\operatorname{div}(X, Y, Z) \quad \text{or} \quad \operatorname{div} R,$$

if R is the resultant of X, Y, Z.

§ 23. **Laplace's and Poisson's equations.**

Let us now identify the vector X, Y, Z with the force of attraction at a point in a gravitational field. For any closed surface in this field not containing matter we have, by the surface integral of normal force theorem, since $lX + mY + nZ = F\cos\theta$,

$$\iint (lX + mY + nZ)\,dS = 0.$$

Hence, by using Gauss's theorem, we obtain

$$\iiint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dx\,dy\,dz = 0.$$

This equation is true no matter what the shape of the surface is. It is therefore true for every element of volume into which the space bounded by the closed surface can be divided, and this can only be true when the integrand itself is zero. Thus, for every point in space devoid of matter,

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0,$$

which becomes $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

on substituting for X, Y and Z their values $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$. The above equation is called Laplace's equation, and is usually written

$$\nabla^2 V = 0,$$

∇^2 being used in this country for the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. On the continent Δ is used for the same operator.

Let us now suppose that a closed surface is drawn in a field containing matter of density ρ, ρ being a function of the coordinates. Then, by the surface integral of normal force theorem,

$$\iint (l\mathsf{X} + m\mathsf{Y} + n\mathsf{Z})\, d\mathsf{S} = -4\pi k\mathsf{M} = -4\pi k \iiint \rho\, dx\, dy\, dz. \qquad \text{v}$$

If X, Y, Z is a continuous function of the coordinates, Gauss's theorem can be applied. Hence

$$\iiint \left(\frac{\partial \mathsf{X}}{\partial x} + \frac{\partial \mathsf{Y}}{\partial y} + \frac{\partial \mathsf{Z}}{\partial z} \right) dx\, dy\, dz = -4\pi k \iiint \rho\, dx\, dy\, dz \qquad \text{vi}$$

or

$$\iiint \left(\frac{\partial \mathsf{X}}{\partial x} + \frac{\partial \mathsf{Y}}{\partial y} + \frac{\partial \mathsf{Z}}{\partial z} + 4\pi k\rho \right) dx\, dy\, dz = 0. \qquad \text{vii}$$

This equation holds no matter what the shape of the closed surface is. Hence, in the same way as before, and on substituting for X, Y and Z, we obtain

$$\frac{\partial^2 \mathsf{V}}{\partial x^2} + \frac{\partial^2 \mathsf{V}}{\partial y^2} + \frac{\partial^2 \mathsf{V}}{\partial z^2} + 4\pi k\rho = 0. \qquad \text{viii}$$

This is Poisson's equation. It holds for a point at which the density is ρ, and includes Laplace's equation as a particular case.

§ 24. Change of coordinates.

In the preceding section Laplace's and Poisson's equations are expressed in cartesian coordinates. For many problems it is necessary to express them in polar or cylindrical coordinates. We can, of course, change from the one system to the other by the methods given in the books on the differential calculus. It is much shorter, however, to use the following physical method.

Suppose that x, y, z, the coordinates of P, are expressed as functions of three new variables ξ, η, ζ. Let us, for example, write

$$x = f_1(\xi, \eta, \zeta), \quad y = f_2(\xi, \eta, \zeta), \quad z = f_3(\xi, \eta, \zeta).$$

If one of the new variables, say ξ, is constant, the point P is restricted to a surface, and its position on that surface is specified by η and ζ. By giving ξ in succession different values we describe an infinite family of surfaces. Similarly, there is an infinite family of η and another infinite family of ζ surfaces. One surface of each family passes through every point in space, and the values of ξ, η, ζ can be regarded as the coordinates of that point. If the three families intersect orthogonally—and this is the only case we have occasion to consider—these coordinates are said to be orthogonal. Examples of orthogonal coordinates are of course cartesians, polars, cylindricals and elliptic coordinates. In cartesians the three families of surfaces are families of planes parallel to the different coordinate planes. In polars we have a family of concentric spheres, a family of cones with the same axis and vertex and a family of planes intersecting in the one straight line. In cylindricals we have a family of coaxal cylinders, a family of

planes intersecting in the axis of the cylinders and a family of planes at right angles to the axis of the cylinders. In elliptic coordinates we have three families of confocal conicoids, ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets.

Let the orthogonal coordinates of P be ξ, η, ζ. Draw the surfaces

$$\xi+\frac{d\xi}{2},\quad \xi-\frac{d\xi}{2},\quad \eta+\frac{d\eta}{2},\quad \eta-\frac{d\eta}{2},\quad \zeta+\frac{d\zeta}{2},\quad \zeta-\frac{d\zeta}{2}.$$

Then these surfaces will mark out a volume element which is approximately rectangular. The length of the side between the $\xi+\frac{d\xi}{2}$ and $\xi-\frac{d\xi}{2}$ surfaces is $\lambda d\xi$, where λ is a function of ξ, η, ζ, and the lengths of the other sides are $\mu d\eta$ and $\nu d\zeta$, μ and ν being also functions of ξ, η and ζ. Let A, B, C be the components of force at P in the ξ, η, ζ directions, that is, normal to the ξ, η, ζ surfaces respectively.

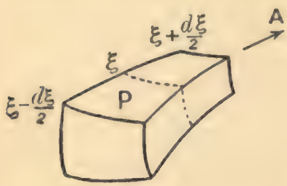

FIG. 15.

Let us find the surface integral of normal force or the total outward flux of force over the surface of the element. The area of the ξ section through P is $\mu\nu\, d\eta\, d\zeta$. Hence the flux of force through this section is $\mathsf{A}\mu\nu\, d\eta\, d\zeta$. The flux inward over the $\xi-\frac{d\xi}{2}$ end is

$$\mathsf{A}\mu\nu\, d\eta\, d\zeta-\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu)\frac{d\xi}{2}\,d\eta\, d\zeta.$$

The flux outward over the $\xi+\frac{d\xi}{2}$ end is

$$\mathsf{A}\mu\nu\, d\eta\, d\zeta+\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu)\frac{d\xi}{2}\,d\eta\, d\zeta.$$

Subtracting, we obtain for the flux outward over these faces

$$\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu)\,d\xi\, d\eta\, d\zeta.$$

By symmetry, taking account of the other faces, we find for the total outward flux of force

$$\left\{\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu)+\frac{\partial}{\partial\eta}(\mathsf{B}\nu\lambda)+\frac{\partial}{\partial\zeta}(\mathsf{C}\lambda\mu)\right\}d\xi\, d\eta\, d\zeta.$$

By the surface integral of normal force theorem this must equal $-4\pi k\mathsf{M}$, where M is the total mass within the element. But

$$\mathsf{M}=\rho\lambda\mu\nu\, d\xi\, d\eta\, d\zeta,$$

where ρ is the density at P. Hence

$$\left\{\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu)+\frac{\partial}{\partial\eta}(\mathsf{B}\nu\lambda)+\frac{\partial}{\partial\zeta}(\mathsf{C}\lambda\mu)\right\}d\xi\,d\eta\,d\zeta=-4\pi k\rho\lambda\mu\nu\,d\xi\,d\eta\,d\zeta,$$

or, on dividing out by $\lambda\mu\nu\,d\xi\,d\eta\,d\zeta$,

$$\frac{1}{\lambda\mu\nu}\left\{\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu)+\frac{\partial}{\partial\eta}(\mathsf{B}\nu\lambda)+\frac{\partial}{\partial\zeta}(\mathsf{C}\lambda\mu)\right\}+4\pi k\rho=0.$$

We have thus proved Poisson's equation for orthogonal coordinates.

Consequently $$\frac{\partial\mathsf{X}}{\partial x}+\frac{\partial\mathsf{Y}}{\partial y}+\frac{\partial\mathsf{Z}}{\partial z}=\frac{1}{\lambda\mu\nu}\Sigma\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu).$$

Since $\lambda d\xi$ denotes a displacement in the ξ direction,

$$\mathsf{A}=\frac{1}{\lambda}\frac{\partial\mathsf{V}}{\partial\xi},\quad \mathsf{B}=\frac{1}{\mu}\frac{\partial\mathsf{V}}{\partial\eta}\quad\text{and}\quad \mathsf{C}=\frac{1}{\nu}\frac{\partial\mathsf{V}}{\partial\zeta}.$$

Hence $$\nabla^2\mathsf{V}=\frac{1}{\lambda\mu\nu}\Sigma\frac{\partial}{\partial\xi}\left(\frac{\mu\nu}{\lambda}\frac{\partial\mathsf{V}}{\partial\xi}\right).$$

It should be noticed that this section affords a means of proving the equations of Laplace and Poisson without using Gauss's theorem.

§ 25. **Poisson's equation in polar and cylindrical coordinates.**

In orthogonal coordinates the equation runs

$$\frac{1}{\lambda\mu\nu}\Sigma\frac{\partial}{\partial\xi}\left(\frac{\mu\nu}{\lambda}\frac{\partial\mathsf{V}}{\partial\xi}\right)+4\pi k\rho=0.$$

To change this into polars we have to write r, θ, ϕ for ξ, η, ζ and 1, r, $r\sin\theta$ for λ, μ, ν, since the sides of the volume element are dr, $r\,d\theta$ and $r\sin\theta\,d\phi$. On making the substitution, we obtain

$$\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\mathsf{V}}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathsf{V}}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\mathsf{V}}{\partial\phi}\right)\right\}+4\pi k\rho=0$$

or $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\mathsf{V}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathsf{V}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathsf{V}}{\partial\phi^2}+4\pi k\rho=0.$$

To change into cylindrical coordinates we have to write r, θ, z for ξ, η, ζ and 1, r, 1 for λ, μ, ν, since the sides of the volume element are now dr, $r\,d\theta$ and dz. On making the substitution, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathsf{V}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\mathsf{V}}{\partial\theta^2}+\frac{\partial^2\mathsf{V}}{\partial z^2}+4\pi k\rho=0.$$

§ 26. **Example on Poisson's equation.**

When ρ is given as a function of the coordinates and the boundary conditions are known, Poisson's equation can be used to determine V.

For example, let the density ρ be a function only of r, the distance from the origin. The attracting matter is distributed therefore in uniform thin concentric spherical shells. We shall also suppose that all the matter is a finite distance from the origin.

It follows from symmetry that V can be a function only of r. Expressing Poisson's equation, therefore, in polar coordinates and putting $\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \phi} = 0$, we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = -4\pi k\rho.$$

On integrating, this becomes

$$r^2\frac{\partial V}{\partial r} = -4\pi k\int_0^r \rho r^2\,dr + C,$$

C being the constant of integration. Since $r^2\frac{\partial V}{\partial r} = 0$ when $r = 0$, $C = 0$.

Hence
$$\frac{\partial V}{\partial r} = -\frac{4\pi k}{r^2}\int_0^r \rho r^2\,dr,$$

and on integrating by parts

$$V = -\int_0^r dr\left\{\frac{4\pi k}{r^2}\int_0^r \rho r^2\,dr\right\} + D = \frac{4\pi k}{r}\int_0^r \rho r^2\,dr - \int_0^r 4\pi k\,\rho r\,dr + D.$$

When r is ∞, $V = 0$, since all the matter is a finite distance from the origin. When r is ∞ the first term on the right-hand side vanishes, since $\int_0^r \rho r^2\,dr$ is finite. Hence $D = \int_0^\infty 4\pi k\rho r\,dr$.

We thus obtain the final value of V, namely,

$$V = \frac{k}{r}\int_0^r 4\pi r^2\rho\,dr + k\int_r^\infty 4\pi r\rho\,dr.$$

But $4\pi r^2\rho\,dr$ is the mass dm of a thin shell of radius r, thickness dr and density ρ, and if m denotes the mass of the whole sphere of radius r,

$$\int_0^r 4\pi r^2\rho\,dr = m.$$

Therefore
$$V = \frac{km}{r} + k\int_{r=r}^{r=\infty}\frac{dm}{r}.$$

The first term represents the part of the potential at a distance r from the origin due to the matter contained within the sphere of radius r. The second term represents the part due to the external shells. For each of these taken singly the interior is an equipotential space, and P has consequently the same potential as on its surface.

§ 27. **Electrical images. Point and plane.**

All the results of the preceding sections hold for electrostatics as well as for gravitational attraction, if we write $k = 1$ and understand by ρ the density of the electrostatic charge. Only, in defining the electrostatic potential at a point, we have to make a condition; when

the unit charge is brought from infinity we suppose it does not disturb the distribution of electricity in the field.

The nature of the problem is, however, in electrostatics somewhat different. Instead of being given the system of charges and asked to calculate the attraction, we are given the conductor or dielectric and have to calculate the distribution of the charge on it, before it is possible to calculate the attraction.

Suppose that we have a charge $+e$ at P and a charge $-e$ at P′ situated a distance $2a$ apart. Let AB be an infinite plane at right angles to PP′ bisecting it. Then the potential on this plane is zero; for if we take any point Q on the plane, the potential at Q is

$$\frac{e}{\mathrm{PQ}} - \frac{e}{\mathrm{P'Q}} \quad \text{and} \quad \mathrm{PQ} = \mathrm{P'Q}.$$

Since this plane is at potential zero, if we suppose it replaced by an infinite thin plate of metal connected with the earth, there will be no alteration of the potential on either side of it, but the field will remain everywhere the same as if it were due solely to the two electric charges P and P′.

If now we keep the metal plate in connection with the earth and remove the charge P′, the potential to the left of AB will become zero, but on the right it will remain the same as before.

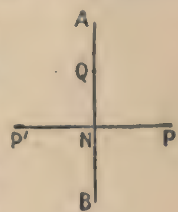

FIG. 16.

Hence if a point charge is placed at P at a distance a from a plane conductor which is at potential zero, the electric field will be that due to the point charge together with that due to an equal and opposite point charge situated at P′, a distance a on the other side of the plane. The charge at P′ is said to be the electrical image of the charge at P.

An electrical image is an electric charge or system of charges on one side of a surface which would produce on the other side of the surface the same electrical action which the actual electrification of the surface really does produce.

Let σ be the surface density of electricity on the plane in the above case of a point charge e at a distance a from an infinite conducting plane at zero potential. To find σ at Q we require the electric field strength at right angles to the plane. The field strength at right angles to the plane at Q due to the charge at P is equal to

$$\frac{e}{\mathrm{PQ}^2}\,\frac{\mathrm{PN}}{\mathrm{PQ}}$$

and acts from right to left. The field strength due to the charge on the plane, being equivalent to that due to the image at P′, has a component in the same direction equal to

$$\frac{e}{\mathrm{P'Q}^2}\,\frac{\mathrm{P'N}}{\mathrm{P'Q}}.$$

Since $P'Q = PQ$ and $P'N = PN$, the resultant field strength at right angles to the plane is

$$\frac{2e}{PQ^2}\frac{PN}{PQ}.$$

The two components parallel to the plane neutralise one another.

Now suppose a small cylinder drawn as in fig. 17, with its ends parallel to the plane AN, the area of each end being dS. Apply the surface integral of normal force theorem to this cylinder. As has been mentioned, there is no force to the left of AN. The end cd contributes a part

$$-\frac{2e}{PQ^2}\frac{PN}{PQ}dS$$

to the integral; the resultant contribution of the side is zero, since the field intensity parallel to the plane is zero. The total quantity of electricity inside the cylinder is σdS. Hence, applying the theorem,

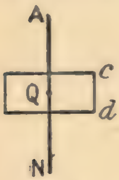

FIG. 17.

$$-\frac{2e}{PQ^2}\frac{PN}{PQ}dS = 4\pi\sigma dS \quad \text{or} \quad \sigma = -\frac{e}{2\pi}\frac{PN}{PQ^3}.$$

As all the tubes of force which start from e end on the plane, the total charge on the latter is $-e$. This may also be found by direct integration.

The force of attraction produced at P by the surface distribution of electricity on the plane is the same as would be produced by the charge $-e$ at P′. The resultant force on the charge at P is hence

$$\frac{e^2}{PP'^2} = \frac{e^2}{4a^2}.$$

§ 28. **Point and sphere.**

Suppose that we have a charge e at P and a charge $-e'$ at P′; e' being less than e. Then the equation to the surface of zero potential is given by

$$\frac{e}{QP} - \frac{e'}{QP'} = 0 \quad \text{or} \quad \frac{QP}{QP'} = \frac{e}{e'}.$$

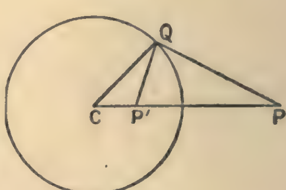

FIG. 18.

At Q make $\angle CQP' = \angle QPP'$ and produce QC to meet PP′ in C.

Then in triangles CQP′, CQP, $\angle QCP$ is common and $\angle CQP' = \angle QPC$. Hence the triangles are similar. Therefore

$$\frac{CP'}{CQ} = \frac{P'Q}{PQ} = \frac{CQ}{CP}.$$

The product of the first and third ratio is equal to the square of the second, that is

$$\frac{CP'}{CP} = \left(\frac{P'Q}{PQ}\right)^2 = \left(\frac{e'}{e}\right)^2.$$

Therefore C is a fixed point.

Also, since $\frac{\mathsf{CP'}}{\mathsf{CQ}} = \frac{\mathsf{CQ}}{\mathsf{CP}}$, $\mathsf{CQ}^2 = \mathsf{CP}\,.\,\mathsf{CP'}$ or CQ is constant. The surface of zero potential is therefore a sphere with centre C.

Denote CQ by a and CP by f.

Suppose now that a thin metal sphere connected with the earth is placed to coincide with the surface of zero potential. It will not affect the field in any way. There are e' tubes of force from the charge e terminating on the outer surface of this sphere, so that it has a charge $-e'$. The inner surface has a charge $+e'$.

Let the charge at P′ vanish; the charge on the inner surface of the sphere will also vanish. The field outside the sphere and the charge on its outer surface remain, however, unaffected.

Hence, if a point charge e be placed at a distance f from the centre of a spherical conductor of radius a, the electric field outside the sphere is that which would be produced by the original charge e together with a negative charge $-e'$ situated on the line joining e to the centre of the sphere at a distance of a^2/f from the latter, e' being equal to ae/f.

The charge $-e'$ at P′ is the image of the charge e at P.

In order to find σ, the density of the charge at Q, we have to find the component of the field intensity along QC. The field intensity at Q due to e is $\frac{e}{\mathsf{PQ}^2}$. Its components are

$$-\frac{\mathsf{QC}}{\mathsf{QP}}\frac{e}{\mathsf{PQ}^2} \text{ along QC} \quad \text{and} \quad +\frac{\mathsf{PC}}{\mathsf{QP}}\frac{e}{\mathsf{PQ}^2} \text{ along PC}.$$

The field intensity at Q due to e' at P′ has components

$$\frac{\mathsf{QC}}{\mathsf{QP'}}\frac{e'}{\mathsf{QP'}^2} \text{ along QC} \quad \text{and} \quad -\frac{\mathsf{P'C}}{\mathsf{QP'}}\frac{e'}{\mathsf{QP'}^2} \text{ along PC}.$$

Now
$$\frac{\mathsf{PC}}{\mathsf{QP}}\frac{e}{\mathsf{PQ}^2} - \frac{\mathsf{P'C}}{\mathsf{QP'}}\frac{e'}{\mathsf{QP'}^2} = \frac{\mathsf{PC}}{\mathsf{QP}}\frac{e}{\mathsf{PQ}^2} - \frac{\mathsf{P'C}}{\mathsf{QP'}^2}\frac{e}{\mathsf{PQ}}$$
$$= \frac{e}{\mathsf{PQ}}\left(\frac{\mathsf{PC}}{\mathsf{PQ}^2} - \frac{\mathsf{P'C}}{\mathsf{P'Q}^2}\right) = 0.$$

The resultant component along PC is therefore zero. The resultant normal component is

$$\frac{\mathsf{QC}e'}{\mathsf{P'Q}^3} - \frac{\mathsf{QC}e}{\mathsf{PQ}^3} = \mathsf{QC}\left(\frac{e}{\mathsf{PQ}\,\mathsf{P'Q}^2} - \frac{e}{\mathsf{PQ}^3}\right) = \frac{e\mathsf{QC}}{\mathsf{PQ}^3}\left(\frac{\mathsf{PQ}^2}{\mathsf{P'Q}^2} - 1\right) = \frac{ea}{\mathsf{PQ}^3}\left(\frac{f^2}{a^2} - 1\right).$$

Hence, in the same way as in § 27, the numerical value of σ is given by

$$\sigma = \frac{1}{\mathsf{PQ}^3}\frac{ea}{4\pi}\left(\frac{f^2}{a^2} - 1\right).$$

It has of course the negative sign.

The point charge is attracted towards the sphere with a force equal to

$$\frac{ee'}{\mathsf{PP'}^2} = \frac{ee'}{\left(f - \frac{a^2}{f}\right)^2} = \frac{e^2a}{f\left(f - \frac{a^2}{f}\right)^2} = \frac{e^2af}{(f^2 - a^2)^2}.$$

EXAMPLES.

1. Find the total charge on the plane in § 27, and on the sphere in § 28, by integrating the expression for the density.

2. In § 27, if the "point charge" is a small charged sphere of radius b, the energy in the field is

$$\frac{1}{2}\frac{e^2}{b}-\frac{1}{4}\frac{e^2}{a}.$$

3. In § 28, if the "point charge" is a small charged sphere of radius b, the energy in the field is

$$\frac{1}{2}\frac{e^2}{b}-\frac{1}{2}\frac{ae^2}{f^2-a^2}.$$

4. In § 28, if the sphere is insulated and without charge instead of being at zero potential, the force on the point charge is an attraction equal to

$$\frac{e^2a^3}{f^3}\frac{2f^2-a^2}{(f^2-a^2)^2}.$$

CHAPTER II.

HYDRODYNAMICS.

§ 29. Hydrodynamics is that part of physics which deals with the motion of fluids. In order to simplify the mathematics the fluids dealt with are usually supposed to be perfect, *i.e.*

(1) they do not support tangential stress,

(2) their structure is continuous.

If any plane surface is immersed in a fluid, according to the first of these assumptions the resultant thrust exerted by the fluid on the surface must be at right angles to it, whether the plane is moving relatively to the fluid or not. In hydrostatics, that is, when the plane is at rest relatively to the fluid, we know as a fact of experience that the thrust is actually at right angles to the plane. In the case of relative motion we know as an experimental fact that the resultant thrust is oblique to the plane and has a component parallel to the plane which resists the relative motion. The definition of viscosity depends on this fact. The first assumption is therefore equivalent to neglecting viscosity.

In deriving the equations of motion, etc., it will be necessary to consider the motion of small fluid elements. According to the second assumption, these elements must still possess the properties of the fluid in bulk. We must never take them so small as to get down to the individual molecules.

There are two methods of treating the motion of a fluid, the "Lagrangian" and the "Eulerian." In the first of these methods we seek to determine the history of every particle of the fluid. In the second we fix our attention on a particular point in space, and attempt to determine the velocity, density and pressure at that point for all times. Here only the Eulerian method will be used.

§ 30. **Acceleration at a point.**

Let u, v, w be the components of the velocity parallel to the coordinate axes at the point x, y, z at the time t. Then u, v, w are functions of x, y, z and t.

Suppose that P is the point x, y, z and that the particle which is at P at the time t moves to the point Q, the coordinates of which are $x + \delta x$, $y + \delta y$, $z + \delta z$ in the interval of time δt.

The acceleration of the particle is the acceleration at P at the time t. The increase in the x component of the velocity of the particle in moving from P to Q is given by

$$\delta u = \frac{\partial u}{\partial t}\delta t + \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z.$$

In going from P to Q we are taking a step forward in time and a step forward in space. The first term on the right is due to the former and the other three to the latter.

The x component of the acceleration of the particle is

$$\underset{\text{Lt}\,\delta t=0}{\frac{\delta u}{\delta t}} = \underset{\text{Lt}\,\delta t=0}{\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial u}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial u}{\partial z}\frac{\delta z}{\delta t}} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

since $\underset{\text{Lt}\,\delta t=0}{\frac{\delta x}{\delta t}} = u, \ldots, \ldots$. Similarly, the y and z components of the acceleration at P are given respectively by

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \quad \text{and} \quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}.$$

Hence if $\frac{d}{dt}$ denote the operator

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

the three components of the acceleration at P can be written

$$\frac{du}{dt},\quad \frac{dv}{dt},\quad \frac{dw}{dt}.$$

§ 31. **Angular velocity at a point.**

Let ρ be the density of the fluid. Consider a small sphere of fluid with its centre at P(x, y, z) and take a point Q$(x+\alpha, y+\beta, z+\gamma)$ close to P inside the sphere. Then the velocity at P is u, v, w and the velocity at Q has the components

$$u + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma\frac{\partial u}{\partial z}, \ \ldots, \ \ldots.$$

The relative velocity of Q to P is therefore

$$\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma\frac{\partial u}{\partial z}, \ \ldots, \ \ldots.$$

The moment of the velocity of Q about an axis through P parallel to the x-axis is

$$\left(\alpha\frac{\partial w}{\partial x} + \beta\frac{\partial w}{\partial y} + \gamma\frac{\partial w}{\partial z}\right)\beta - \left(\alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y} + \gamma\frac{\partial v}{\partial z}\right)\gamma.$$

Multiply the above expression by $\rho\,dx\,dy\,dz$ and integrate throughout the sphere, and we shall obtain the angular momentum of the sphere about an axis through its centre parallel to the x-axis. The sphere is so small that the differential coefficients $\frac{\partial w}{\partial x}$ etc. may be regarded as constant throughout the field of integration. The product terms such as $\iiint \alpha\beta\,dx\,dy\,dz$ vanish because to every positive value of $\alpha\beta$ there corresponds an equal negative one. Also, by symmetry,

$$\iiint \beta^2\,dx\,dy\,dz = \iiint \gamma^2\,dx\,dy\,dz.$$

Hence the angular momentum of the sphere about an axis through its centre parallel to Ox is given by

$$\iiint\left(\beta^2\frac{\partial w}{\partial y} - \gamma^2\frac{\partial v}{\partial z}\right)\rho\,dx\,dy\,dz = \iiint \beta^2\rho\,dx\,dy\,dz\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right).$$

But the moment of inertia of the sphere about this axis is

$$\iiint(\beta^2+\gamma^2)\rho\,dx\,dy\,dz = 2\iiint \beta^2\rho\,dx\,dy\,dz,$$

and hence the angular velocity about this axis is

$$\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right).$$

Let the components of angular velocity at the point P be denoted by ξ, η, ζ. Then

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),\quad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),\quad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

§ 32. **Curl of a vector. Potential vectors. Stream lines.**

The vector, the components of which are

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

is said to be the curl of the vector, the components of which are u, v, w. Hence the angular velocity at any point is half the curl of the velocity at that point.

Consider the expression

$$-d\phi = u\,dx + v\,dy + w\,dz.$$

If the curl is zero, $d\phi$ is a perfect differential,

$$u = -\frac{\partial \phi}{\partial x},\quad v = -\frac{\partial \phi}{\partial y},\quad w = -\frac{\partial \phi}{\partial z},$$

the vector u, v, w is derivable from a potential, and is said to be a **potential vector.**

When u, v, w denote the velocity at a point, ϕ is called the velocity potential. Obviously, the condition for the existence of a velocity potential is that the motion should be irrotational.

If we define stream lines for any given time, as curves the tangents to which everywhere give the direction in which the fluid is moving at that time, then the equation to the stream lines is

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}.$$

The stream lines obviously cut the surfaces given by $\phi = \text{constant}$ at right angles, and the stream lines and velocity potential have the same properties as the lines of force and force potential in the theory of attraction. One point, however, calls for attention, the convention about the sign of the potential is different. In attraction we had

$$X=\frac{\partial V}{\partial x}.$$

§ 33. **Equation of continuity**.

Consider a rectangular fluid element, the centre of which is $P(x, y, z)$ and the sides of which are dx, dy, dz. Let ρ as usual be the density of the fluid at P and let u, v, w be the components of the velocity at P.

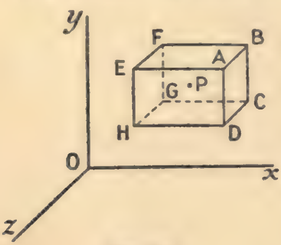

Fig. 19.

The average value of ρu on the face EFGH is $\rho u-\frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$. Hence the rate at which fluid enters the element through this face is

$$\left(\rho u-\frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dy\,dz. \quad \ldots\ \ldots.(1)$$

The average value of ρu on the face ABCD is $\rho u+\frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$. Hence the rate at which fluid leaves the element through this face is

$$\left(\rho u+\frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dy\,dz. \quad \ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By subtracting (2) from (1), we find that the rate at which matter is being gained through these faces is

$$-\frac{\partial(\rho u)}{\partial x}\,dx\,dy\,dz.$$

Similarly, the rates at which matter is being gained through the faces parallel to zx and xy are respectively

$$-\frac{\partial(\rho v)}{\partial y}\,dx\,dy\,dz \quad \text{and} \quad -\frac{\partial(\rho w)}{\partial z}\,dx\,dy\,dz.$$

The total rate at which matter is entering through the surface of the element is therefore

$$-\left\{\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right\}dx\,dy\,dz. \quad \text{.................(3)}$$

The mass of the element is $\rho\,dx\,dy\,dz$ and the rate at which it is increasing is

$$\frac{\partial\rho}{\partial t}dx\,dy\,dz. \quad \text{.................................(4)}$$

Hence, equating (3) and (4), we obtain

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0, \quad \text{....................(5)}$$

the equation of continuity in its most general form in cartesian coordinates.

It is obvious that (5) may always be written in the form

$$\frac{d\rho}{dt}+\rho\operatorname{div} q=0,$$

q being the resultant of u, v and w.

If the liquid is incompressible and homogeneous, ρ is constant, and if the motion is irrotational, $u=-\frac{\partial\phi}{\partial x}$ etc. Hence the equation of continuity reduces to

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad \text{or} \quad \nabla^2\phi=0.$$

This is the same as Laplace's equation.

§ 34. **Equation of continuity in polar and cylindrical coordinates.**

The equation of continuity in cylindrical and spherical polar coordinates may be derived directly from first principles by considering the rate of flow into an element and the rate at which the mass of the element increases. Or it may be derived in the same way in the first instance in generalised orthogonal coordinates and the transition afterwards made to polars or cylindricals.

We shall here assume the result proved in § 24, namely that

$$\frac{\partial \mathsf{X}}{\partial x}+\frac{\partial \mathsf{Y}}{\partial y}+\frac{\partial \mathsf{Z}}{\partial z}=\frac{1}{\lambda\mu\nu}\Sigma\frac{\partial}{\partial\xi}(\mathsf{A}\mu\nu), \quad \text{.......................(6)}$$

X, Y, Z being the components of a vector in the x, y, z directions, ξ, η, ζ generalised orthogonal coordinates, $\lambda\,d\xi$, $\mu\,d\eta$, $\nu\,d\zeta$ the lengths of the sides of the volume element bounded by ξ, η, ζ surfaces and A, B, C the components of the vector in the ξ, η, ζ directions.

To obtain the equation in polars, write ρu, ρv, ρw, $\rho\mathsf{U}$, $\rho\mathsf{V}$, $\rho\mathsf{W}$, r, θ, ϕ, 1, r, $r\sin\theta$ for X, Y, Z, A, B, C, ξ, η, ζ, λ, μ, ν, and substitute in (6). Then, by means of (5),

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(\rho\mathsf{U}r^2\sin\theta)+\frac{\partial}{\partial\theta}(\rho\mathsf{V}r\sin\theta)+\frac{\partial}{\partial\phi}(\rho\mathsf{W}r)\right\}=0. \quad \text{....(7)}$$

Here U, V, W are the components of the velocity at P in the r, θ, ϕ directions. If the motion is irrotational, $\mathsf{U} = -\frac{\partial \Phi}{\partial r}$, $\mathsf{V} = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}$ and $\mathsf{W} = -\frac{1}{r \sin\theta}\frac{\partial \Phi}{\partial \phi}$, Φ being used as velocity potential to prevent confusion with the coordinate ϕ.

To obtain the equation in cylindricals, write r, θ, z for ξ, η, ζ and 1, r, 1 for λ, μ, ν. Then we obtain

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left\{\frac{\partial}{\partial r}(\rho \mathsf{U} r) + \frac{\partial}{\partial \theta}(\rho \mathsf{V}) + \frac{\partial}{\partial z}(\rho \mathsf{W} r)\right\} = 0. \quad \text{..............(8)}$$

Here U, V, W are the components of the velocity at P in the r, θ, z directions, and if the motion is irrotational, $\mathsf{U} = -\frac{\partial \phi}{\partial r}$, $\mathsf{V} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$ and $\mathsf{W} = -\frac{\partial \phi}{\partial z}$.

§ 35. Particular cases can be derived from the general equations (7) and (8), but it is better to obtain simple cases from first principles. For example, suppose we have steady radial flow of an incompressible liquid from a point. Take the point as origin and take as volume element a shell bounded by spheres of radii r and $r + dr$ and with the origin as centre. If v denote the outward velocity at distance r, the quantity of liquid entering the element per second is $4\pi r^2 v$, and the quantity leaving it per second is $4\pi r^2 v + 4\pi \frac{\partial}{\partial r}(r^2 v)\,dr$. These are equal. Hence the equation of continuity takes the form

$$\frac{\partial}{\partial r}(r^2 v) = 0.$$

§ 36. **Equations of motion.**

Consider, as before, a rectangular fluid element with centre at $\mathsf{P}(x, y, z)$ and sides dx, dy, dz. Let u, v, w be the velocity, p the pressure and ρ the density at P, and let X, Y, Z be the components of external force per unit mass at P.

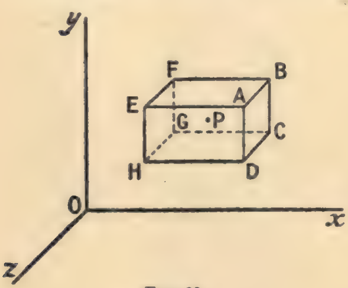

FIG. 20.

The rate of change of momentum of the element in the x direction is

$$\rho\, dx\, dy\, dz \frac{du}{dt}. \quad \text{.........(9)}$$

The "body force" on the element in that direction is

$$\rho \mathsf{X}\, dx\, dy\, dz. \quad \text{...........(10)}$$

The average pressure on face EFGH is

$$p - \frac{\partial p}{\partial x}\frac{dx}{2},$$

hence the thrust on that face is

$$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)dy\,dz$$

and the thrust on face ABCD is

$$\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)dy\,dz.$$

The resultant pressure-thrust in the x direction is consequently

$$-\frac{\partial p}{\partial x}dx\,dy\,dz. \quad \text{.............................(11)}$$

Equating (9) to the sum of (10) and (11) and cancelling out $dx\,dy\,dz$, we find the x equation of motion

$$\rho\frac{du}{dt} = \rho X - \frac{\partial p}{\partial x} \quad \text{or} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}. \quad \text{.....(12)}$$

The similar y and z equations are

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}.$$

§ 37. Case of impulsive pressure.

It is possible that we may have an extremely great pressure acting for a very short time. The diagram illustrating the hydrostatic paradox, fig. 21, is a case in point. P and Q are two pistons working in cylinders fitted into a vessel containing an incompressible liquid. If the one piston be driven in smartly by a blow from a hammer, an impulsive pressure is transmitted throughout the liquid

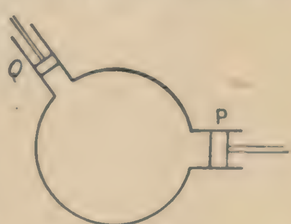

FIG. 21.

Let ω be the impulsive pressure at the point x, y, z inside the liquid, *i.e.* $\omega = \int p\,dt$ for the point taken over the interval of time τ through which the impulsive pressure acts. Let u, v, w be the component velocities at the point before, and u', v', w' the component velocities immediately after the impulse.

Consider as before a rectangular fluid element with its centre at x, y, z. The increase of momentum produced in the x direction is

$$(u' - u)\rho\,dx\,dy\,dz.$$

The impulsive thrust in the x direction is

$$\left(\omega - \frac{\partial \omega}{\partial x}\frac{dx}{2}\right) dy\, dz$$

and in the $-x$ direction $\left(\omega + \frac{\partial \omega}{\partial x}\frac{dx}{2}\right) dy\, dz.$

Hence the resultant impulsive thrust in the x direction is

$$-\frac{\partial \omega}{\partial x}\, dx\, dy\, dz.$$

The equation for the impulsive generation of motion is therefore

Similarly
$$\left.\begin{aligned} u' - u &= -\frac{1}{\rho}\frac{\partial \omega}{\partial x}. \\ v' - v &= -\frac{1}{\rho}\frac{\partial \omega}{\partial y}, \\ w' - w &= -\frac{1}{\rho}\frac{\partial \omega}{\partial z}. \end{aligned}\right\} \qquad (13)$$

The effect of finite forces during τ is of course neglected.

Terms may be added to (13) to represent possible impulsive body forces acting on the liquid, but such forces are only of mathematical interest.

An interesting result follows from equations (13). If the first, second and third be differentiated respectively with regard to x, y and z, and if the right-hand sides and left-hand sides of the equations thus formed be added, we obtain

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{1}{\rho}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2}\right).$$

The left-hand side of this equation vanishes since the fluid is incompressible, and the equation consequently reduces to

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0.$$

Impulsive pressure therefore satisfies the same equation as gravitational potential and is transmitted instantaneously to all parts of an incompressible fluid.

§ 38. **Boundary condition.**

At a fixed boundary the normal component of the velocity must be zero. Hence

$$lu + mv + nw = 0,$$

where l, m, n give the direction cosines of the normal to the fixed boundary, the direction outwards from the fluid being positive.

If the boundary is moving with velocity V in a direction making an angle θ with the outward drawn normal from the fluid, then we must have

$$lu + mv + nw = \mathsf{V} \cos \theta.$$

If the motion is irrotational, this becomes

$$-\frac{\partial\phi}{\partial n}=\mathsf{V}\cos\theta,$$

$\frac{\partial}{\partial n}$ denoting a differentiation in the direction l, m, n.

§ 39. **Green's theorem.**

According to Gauss's theorem, § 21,

$$\iiint\left(\frac{\partial \mathsf{X}}{\partial x}+\frac{\partial \mathsf{Y}}{\partial y}+\frac{\partial \mathsf{Z}}{\partial z}\right)dx\,dy\,dz=\iint(l\mathsf{X}+m\mathsf{Y}+n\mathsf{Z})\,d\mathsf{S},$$

where the volume integral is taken throughout a certain region of space and the surface integral is taken over the surface bounding the same region, l, m, n being positive outwards and X, Y, Z being a continuous function of the coordinates.

For X, Y, Z write pu, pv, pw, where p is a scalar and u, v, w a vector function of the coordinates. Then Gauss's theorem becomes

$$\iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dx\,dy\,dz+\iiint\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)dx\,dy\,dz$$
$$=\iint p(lu+mv+nw)\,d\mathsf{S}.$$

This equation is a special form of Green's theorem.

§ 40. **Energy equation.**

Suppose that X, Y, Z, the external force per unit mass of the fluid, can be derived from a potential Ω which is independent of the time. Then

$$\mathsf{X}=-\frac{\partial\Omega}{\partial x},\quad \mathsf{Y}=-\frac{\partial\Omega}{\partial y},\quad \mathsf{Z}=-\frac{\partial\Omega}{\partial z},\quad \frac{\partial\Omega}{\partial t}=0,$$

and the equations of motion can be written

$$\rho\frac{du}{dt}=-\rho\frac{\partial\Omega}{\partial x}-\frac{\partial p}{\partial x},\quad \rho\frac{dv}{dt}=-\rho\frac{\partial\Omega}{\partial y}-\frac{\partial p}{\partial y},\quad \rho\frac{dw}{dt}=-\rho\frac{\partial\Omega}{\partial z}-\frac{\partial p}{\partial z}.\quad \ldots(14)$$

Multiply the first, second and third of the above equations by u, v and w respectively, and add the right-hand and left-hand sides of the three equations thus formed. We obtain as a result the equation

$$\rho\left(u\frac{du}{dt}+v\frac{dv}{dt}+w\frac{dw}{dt}\right)=-\rho\left(u\frac{\partial\Omega}{\partial x}+v\frac{\partial\Omega}{\partial y}+w\frac{\partial\Omega}{\partial z}\right)-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right),$$

which simplifies to

$$\frac{1}{2}\rho\frac{d}{dt}(u^2+v^2+w^2)+\rho\frac{d\Omega}{dt}=-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right).$$

Assume that the density ρ is constant, multiply by $dx\,dy\,dz$ and integrate throughout all the region of space occupied by the fluid. This gives

$$\frac{1}{2}\iiint \frac{d}{dt}\{\rho(u^2+v^2+w^2)\}\,dx\,dy\,dz+\iiint\frac{d}{dt}(\rho\Omega)\,dx\,dy\,dz$$
$$=-\iiint\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)dx\,dy\,dz. \quad\text{.........(15)}$$

Now, by Green's theorem,

$$\iiint\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)dx\,dy\,dz=\iint p\,(lu+mv+nw)\,dS,$$

the other term vanishing, because the continuity equation takes the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

owing to ρ being constant. Also, the order of the integration and differentiation in the first two terms of equation (15) is clearly interchangeable, and T, the total kinetic energy, and V, the total potential energy of the liquid in the region, are given by

$$\mathsf{T}=\frac{1}{2}\iiint\rho\,(u^2+v^2+w^2)\,dx\,dy\,dz,\quad \mathsf{V}=\iiint\rho\Omega\,dx\,dy\,dz.$$

Making these substitutions in (15), we obtain

$$\frac{d}{dt}(\mathsf{T}+\mathsf{V})=-\iint p\,(lu+mv+nw)\,dS, \quad\text{................(16)}$$

that is, the rate at which the energy of the region is increasing is equal to the rate at which pressure forces are doing work upon its surface.

If the motion is irrotational, the right-hand side of (16) becomes

$$\iint p\frac{\partial\phi}{\partial n}\,dS.$$

§ 41. **Integration of the equations of motion.**

The equations of motion can be integrated whenever a velocity potential exists, if the forces are derivable from a potential and if ρ is a function of the pressure only. Rewriting them from p. 35 for the sake of convenience, we have

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=\mathsf{X}-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=\mathsf{Y}-\frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=\mathsf{Z}-\frac{1}{\rho}\frac{\partial p}{\partial z}.\end{aligned}\right\}\quad\text{...............(12)}$$

If the motion is irrotational,

$$\frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}.$$

Substituting these results in (12) and at the same time writing $u=-\frac{\partial \phi}{\partial x}$, $v=-\frac{\partial \phi}{\partial y}$, $w=-\frac{\partial \phi}{\partial z}$ in the first term of these equations, we obtain

$$-\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=\mathsf{X}-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$-\frac{\partial}{\partial t}\frac{\partial \phi}{\partial y}+u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}=\mathsf{Y}-\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$-\frac{\partial}{\partial t}\frac{\partial \phi}{\partial z}+u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}=\mathsf{Z}-\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

Let q be the resultant of u, v, w and let X, Y, Z be derivable, as in § 40, from a potential which is independent of the time. Then substitute, and the equations become

$$-\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}q^2\right)=-\frac{\partial \Omega}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$-\frac{\partial}{\partial y}\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial y}\left(\frac{1}{2}q^2\right)=-\frac{\partial \Omega}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$-\frac{\partial}{\partial z}\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial z}\left(\frac{1}{2}q^2\right)=-\frac{\partial \Omega}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

Multiply respectively by dx, dy, dz and add; then, for any definite value of t,

$$-d\left(\frac{\partial \phi}{\partial t}\right)+d\left(\frac{1}{2}q^2\right)=-d\Omega-\frac{dp}{\rho}.$$

This has the integral

$$\int\frac{dp}{\rho}=\frac{\partial \phi}{\partial t}-\Omega-\frac{1}{2}q^2+\mathsf{F}(t), \quad \ldots\ldots\ldots\ldots(17)$$

since the constant of integration is a function of t. Since ϕ is indeterminate to an additive function of t, we may suppose $\mathsf{F}(t)$ included in $\frac{\partial \phi}{\partial t}$, in which case the last term of the equation vanishes.

In steady motion, when the fluid is incompressible, the equation becomes

$$\frac{p}{\rho}=-\Omega-\frac{1}{2}q^2+\mathsf{C}, \quad \ldots\ldots\ldots\ldots(18)$$

C being a constant.

§ 42. **Bernoulli's theorem.**

Let the motion be steady and let the distance along a stream line be denoted by s. Then the acceleration along the stream line, at any point on it, is given by

$$q\frac{\partial q}{\partial s},$$

q of course denoting the velocity along the stream line. By analogy with the right hand of equation (14), the resultant force per unit mass at the point in the direction of the stream line is

$$-\frac{\partial \Omega}{\partial s}-\frac{1}{\rho}\frac{\partial p}{\partial s},$$

whence $$\frac{1}{\rho}\frac{\partial p}{\partial s}=-\frac{\partial \Omega}{\partial s}-q\frac{\partial q}{\partial s} \quad \text{and} \quad \int\frac{dp}{\rho}=-\Omega-\frac{1}{2}q^2+\mathsf{D}. \quad \text{.........(19)}$$

D being a constant of integration.

Bernoulli employed a different method of proof. We proceed to give his proof for the case when the fluid is incompressible.

Consider a portion of a tube of flow, *i.e.* an infinitely narrow tube the surface of which consists of stream lines, and denote the positions of the ends, which are normal to the stream lines, by A and B. Let the direction of flow be from A to B.

FIG. 22.

Let the velocity, pressure, cross-sectional area and force potential at B and A be denoted respectively by q, p, σ, Ω and q', p', σ', Ω'. In each unit of time a mass $q'\sigma'\rho$ enters at A and an equal mass $q\sigma\rho$ leaves at B. Hence

$$q'\sigma'=q\sigma.$$

The mass entering per unit of time at A brings with it the energy

$$q'\sigma'\rho(\tfrac{1}{2}q'^2+\Omega'),$$

while the mass leaving per unit of time at B takes with it the energy

$$q\sigma\rho(\tfrac{1}{2}q^2+\Omega).$$

The work done per unit of time by the pressure on the mass entering at A is

$$p'\sigma'q',$$

while the work done per unit of time on the mass leaving at B is

$$p\sigma q.$$

Hence, since the energy in the tube is constant,

$$p'\sigma'q'+q'\sigma'\rho(\tfrac{1}{2}q'^2+\Omega')=p\sigma q+q\sigma\rho(\tfrac{1}{2}q^2+\Omega),$$

which simplifies to

$$\frac{p}{\rho}+\Omega+\frac{1}{2}q^2=\mathsf{D}. \quad \text{.............. .... (20)}$$

It should be noted that this equation is not the same as equation (18) of the preceding section. This equation holds for rotational motion

and D is constant only so long as we keep to one stream line. Equation (18) holds only for irrotational motion, but C is constant throughout the field.

§ 43. **Applications of Bernoulli's theorem.**

First take the case of liquid flowing in a horizontal pipe of varying cross-section. The velocity is greatest where the cross-sectional area is smallest. Apply (20). Since the pipe is horizontal, Ω may be taken as approximately constant along the stream lines; hence, when q is greatest, p is least. The pressure must therefore be least at the narrow parts, as is shown by the gauge tubes.

Next consider the case of the efflux of a liquid from a small hole in the side of a vessel, which is kept filled up to a constant level. Then the motion is steady. Take the origin in the surface of the liquid and the axis of z vertically downwards. Then $\Omega = -gz$.

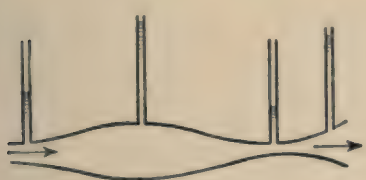

FIG. 23.

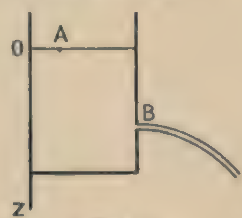

FIG. 24.

Take a stream line which is on the surface of the jet at B. It may be supposed to start from the surface of the liquid at A. To determine the constant D for the stream line, substitute the values for p, Ω and q at A. The velocity at the surface may be supposed to be zero; hence $q=0$. Also $\Omega=0$ and $p=\Pi$, the atmospheric pressure. Hence $D=\Pi/\rho$, and throughout the stream line

$$\frac{p}{\rho}+\Omega+\frac{1}{2}q^2=\frac{\Pi}{\rho}.$$

At B we have $p=\Pi$ and $\Omega = -gz$. Hence the velocity at B is given by $q^2=2gz$.

This result is known as Torricelli's theorem.

It is a matter of experience that the jet, when it issues, is not cylindrical in form. The stream lines converge inside the vessel, and this convergence continues until a point outside is reached, where the cross-sectional area of the jet is a minimum. This point is called the *vena contracta*. At the vena contracta the jet is approximately cylindrical. The area of the vena contracta depends on the nature of the hole and on whether it is fitted with a mouthpiece or not; in the case of a simple hole in a thin wall the area of the vena contracta is found by experiment to be about ·62 times the area of the hole.

Owing to the curvature of the stream lines the pressure is not the same throughout any cross-section of the jet except at the vena contracta. Consequently only there will the velocity be uniform and only there is it given throughout the cross-section of the jet by Torricelli's theorem. We cannot therefore calculate the rate of efflux of the liquid unless we know the area of the vena contracta.

EXAMPLES.

1. Fluid is moving in a fine tube of variable section K; prove that the equation of continuity is

$$\frac{\partial}{\partial t}(K\rho)+\frac{\partial}{\partial s}(K\rho v)=0,$$

where v is the velocity at the point s.

2. Find the equation of continuity in a form suitable for air in a tube, and prove that if the density be $f(at-x)$, where t is the time and x the distance from one end of a uniform tube, the velocity is

$$\frac{af(at-x)+(V-a)f(at)}{f(at-x)},$$

where V is the velocity at that end of the tube.

3. If $F(x, y, z, t)$ is the equation of a moving surface, the velocity of the surface normal to itself is

$$-\frac{1}{R}\frac{dF}{dt}, \quad \text{where} \quad R^2=\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2$$

(proved in Lamb's *Hydrodynamics*, p. 7).

4. Establish the differential equation for the equilibrium of a fluid, namely,

$$dp=\rho(X\,dx+Y\,dy+Z\,dz).$$

A vertical cylinder of gas (section A and height h) has mass M at uniform temperature. If k denote p_0/ρ_0, where p_0 and ρ_0 are corresponding pressure and density, prove that the density ρ at depth z below the top of the cylinder is given by $\rho=Ce^{gz/k}$, where C is a constant to be found in terms of A, h, k and M.

5. If the velocity potential is of the form $\phi=z\log r$, and if the density at a point fixed in space is independent of the time, show that the surfaces of equal density are of the form

$$r^2(\log r-\tfrac{1}{2})=z^2+f(\theta\rho),$$

where ρ is the density and r, θ, z are cylindrical coordinates.

6. If a liquid is in equilibrium and the components of the force per unit mass acting on it are

$$X=y^2+yz+z^2, \quad Y=z^2+zx+x^2, \quad Z=x^2+xy+y^2,$$

show that the density at x, y, z must be of the form

$$F\left(\frac{yz+zx+xy}{x+y+z}\right)\Big/(x+y+z)^2,$$

where F is an arbitrary function. (Hint: the density is the integrating factor that makes $X\,dx+Y\,dy+Z\,dz$ a perfect differential.)

7. If an incompressible fluid is at rest under the action of a system of forces, show that they must be derivable from a potential.

8. A liquid is in equilibrium under the action of forces $X=\mu(y+z)$, $Y=\mu(z+x)$, $Z=\mu(x+y)$; show that the surfaces of equal pressure are hyperboloids of revolution.

9. Show that if a fluid moves about an axis so that the stream lines are circles, a velocity potential will exist if the velocity be inversely proportional to the distance from the axis. Hence prove, that if the axis be vertical and the fluid be acted on by gravity the equation of a surface of equal pressure is $r^2z=c$, where r is the distance of a point from the axis, z its distance from a fixed horizontal plane and c is a constant.

10. A right circular cylinder of radius a, closed by plane faces perpendicular to its axis, is filled with liquid. The axis Oz is the axis of the cylinder and the liquid is acted on by external forces whose x and y components per unit mass are $Ax+By$, $Cx+Dy$ respectively.

Prove that the liquid will rotate as a whole about Oz with uniform angular acceleration $\frac{1}{2}(C-B)$, and if the pressure at the origin is zero and ω is the angular velocity of the liquid, show that the resultant force on each plane end of the cylinder is
$$\tfrac{1}{4}\pi\rho a^4\{\omega^2+\tfrac{1}{2}(A+D)\}.$$

11. Apply Green's theorem to show that the problem of finding an irrotational motion of an incompressible fluid, which has prescribed values of normal velocity at the boundaries, admits but one solution.

12. Assuming that the equations of motion of liquid in a rotating ellipsoidal shell (equation $x^2/a^2+y^2/b^2+z^2/c^2=1$) can be expressed in the form
$$\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V}{\partial x}+\alpha x+hy+gz=0,$$
$$\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial V}{\partial y}+hx+\beta y+fz=0,$$
$$\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\partial V}{\partial z}+gx+fy+\gamma z=0,$$
so that the component space accelerations are $\alpha x+hy+gz$, $hx+\beta y+fz$, $gx+fy+\gamma z$, show that, if the forces represented by $-\partial V/\partial x$, $-\partial V/\partial y$, $-\partial V/\partial z$ be only those due to the mutual attraction of the parts of the liquid, the principle of constancy of angular momentum gives $f=g=h=0$.

Hence, taking $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z=Ax$, By, Cz, show by integration that the surfaces of equal pressure are similar coaxial surfaces, and are similar to the containing surface if
$$(A+\alpha)a^2=(B+\beta)b^2=(C+\gamma)c^2,$$
so that the external case might be removed.

13. Prove that the accelerations parallel to the axes can be written in the form
$$\frac{\partial u}{\partial t}+\operatorname{div}\{u(u,\ v,\ w)\}-u\operatorname{div}(u,\ v,\ w)$$
with two similar equations. (By the expression $u(u,\ v,\ w)$ is meant u^2, uv, uw.)

Prove that if q be the resultant velocity of the fluid at any point and ds be an element of path in the direction of flow, while q' is the velocity in any other direction at the same point, the acceleration in this latter direction at the point can be written
$$\frac{\partial q'}{\partial t}+q\frac{\partial q'}{\partial s}.$$

§ 44. Two-dimensional motion. The stream function.

If w is zero, and if u, v are functions of x, y only, then the motion takes place in planes parallel to the xy-plane and is the same in every one of these planes. When we know the motion for the plane $z=0$, we know it everywhere. Consequently this case is said to be one of two-dimensional motion; for analytically it is the same as if the motion were confined to an infinitely thin layer. When we speak of two-dimensional motion in what follows, we shall be understood to refer to the above case; when, for the sake of convenience, we refer to points and curves in the plane $z=0$, we shall understand the lines through the points parallel to Oz and the surfaces parallel to Oz, of which the curves are the traces. Finally, when we refer to liquid flowing across a curve in the plane $z=0$, we shall understand the quantity of liquid that flows through that portion of the cylindrical surface parallel to Oz, which has the curve as base, comprised between the planes $z=0$ and $z=1$.

Let OP (full line) be any curve through the origin in the xy-plane and let ψ denote the quantity of liquid, supposed of unit density, that flows across OP per unit of time, from right to left. Then

$$\psi = \int_0^P (lu + mv)\, ds,$$

where l, m are the direction cosines of the normal to the element ds, the normal being drawn to the left of the curve.

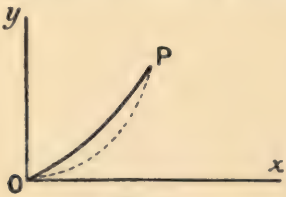

Fig. 25.

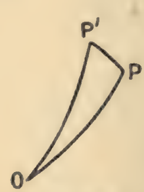

Fig. 26.

Let the liquid be incompressible. Then ψ is a function only of x, y, the coordinates of P. For if any curve represented by the dotted line be drawn joining OP, ψ must be the same for both curves, since the quantity of fluid contained between the two curves remains constant. Similarly it must be the same for all possible curves drawn between O and P. Hence it does not depend on the shape of OP, but only on the position of its end point P, and is thus a function only of the coordinates of that point.

Let P move from P to P′ along a stream line. Since PP′ is a portion of a stream line, no liquid flows across it, and ψ has the same value for P′ that it has for P. Hence

$$\psi = \text{const.}$$

is the equation of a stream line. If we were to shift the reference point O, the only result would be to add a constant quantity to all the expressions for ψ. We may therefore regard ψ as indeterminate to the extent of an additive constant.

Let AB be an element of length ds on the curve OP. Then AC $= dx$, CB $= dy$. From fig. 27 it is obvious that

$$l = -\cos \mathrm{ABC} \quad \text{and} \quad m = \cos \mathrm{BAC}.$$

Hence $dy = -l\,ds$, $dx = m\,ds$ and

$$\psi = \int_0^P (lu + mv)\,ds = -\int u\,dy + \int v\,dx \quad \text{or} \quad d\psi = v\,dx - u\,dy.$$

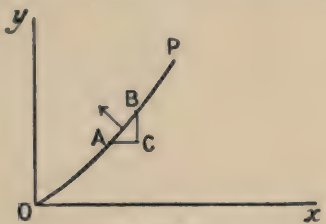

FIG. 27.

But since ψ is a function of the coordinates,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

Hence $v = \frac{\partial \psi}{\partial x}$, $u = -\frac{\partial \psi}{\partial y}$. These expressions hold whether the motion is irrotational or not.

They might have been obtained otherwise. For two-dimensional motion in an incompressible liquid the equation of continuity takes the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and this is the condition that

$$v\,dx - u\,dy$$

is a perfect differential. We have only to put this equal to $d\psi$ and the expressions follow.

Now suppose that the motion is irrotational. Then

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \phi}{\partial y} = +\frac{\partial \psi}{\partial x},$$

and consequently

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = 0.$$

The two families of curves, ϕ = constant and ψ = constant, intersect orthogonally. This again might have been inferred from the definition of ψ.

§ 45. **Expression for the kinetic energy.**

By Green's theorem

$$\iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dx\,dy\,dz+\iiint\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)dx\,dy\,dz$$
$$=\iint p\,(lu+mv+nw)\,d\mathsf{S}.$$

On substituting $p=\phi$, $u=-\dfrac{\partial\phi}{\partial x}$, $v=-\dfrac{\partial\phi}{\partial y}$, $w=-\dfrac{\partial\phi}{\partial z}$, this becomes

$$\iiint\phi\nabla^2\phi\,dx\,dy\,dz+\iiint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}dx\,dy\,dz$$
$$=\iint\phi\left(l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}+n\frac{\partial\phi}{\partial z}\right)d\mathsf{S}.$$

The kinetic energy, T, of the fluid in the volume through which the integration is taken, the motion being irrotational, is given by

$$2\mathsf{T}=\rho\iiint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}dx\,dy\,dz.$$

The density is here assumed to be constant. Hence, by the equation of continuity,

$$\nabla^2\phi=0\quad\text{and}\quad 2\mathsf{T}=\rho\iint\phi\frac{\partial\phi}{\partial n}d\mathsf{S},$$

$\dfrac{\partial}{\partial n}$ denoting a differentiation in the direction of the outward drawn normal.

Let us assume now that we are dealing with two-dimensional motion. Then $\dfrac{\partial\phi}{\partial z}=0$, and the volume reduces to a cylinder with its generators parallel to $\mathsf{O}z$, the ends being given by the planes $z=0$ and $z=1$. The kinetic energy is given by

$$2\mathsf{T}=\rho\iint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}dx\,dy,$$

a surface integral taken over an end of the cylinder.

The surface integral $\displaystyle\iint\phi\frac{\partial\phi}{\partial n}d\mathsf{S}$ reduces to $\displaystyle\int\phi\frac{\partial\phi}{\partial n}ds$, a line integral taken round the trace of the cylinder on the plane $z=0$.

Let l, m be the direction cosines of the outward drawn normal to this curve. Then

$$\frac{\partial\phi}{\partial n}=l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}=l\frac{\partial\psi}{\partial y}-m\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial s},$$

since $-m$, l are the direction cosines of ds, that is, of the tangent to the curve. The positive direction of s is that in which, on going round the curve, the area enclosed by it is on the left hand.

In two-dimensional motion, therefore, the kinetic energy is given by

$$2T = \rho\int\phi\frac{\partial\phi}{\partial n}\,ds = \rho\int\phi\,d\psi.$$

§ 46. **Conjugate functions.**

Two real variables are necessary to specify the position of a point P in a plane. One complex variable $x+iy$ is sufficient, containing as it does within itself both a real x and a real y which can be laid off along their respective axes.

If $x+iy$ be put equal to $re^{i\theta}$, r, which is represented by OP in the diagram, is said to be the modulus of the complex variable, and the angle θ, represented by xOP, is said to be its amplitude.

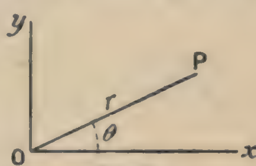

Fig. 28.

A function of both variables x and y is said to be a function of $x+iy$ when it has a differential coefficient with respect to the latter. For example, $A(x+iy)^3+B(x+iy)$ and $\sin a(x+iy)$ are functions of $x+iy$, while Ax^3+Biy and $\sin(ax+iy)$ are not.

Now take any function of a complex variable, and separate its real and imaginary parts, *i.e.* let

$$\phi+i\psi=f(x+iy), \quad\ldots\ldots\ldots\ldots(21)$$

where ϕ and ψ are both real functions of x and y. We have

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=f'(x+iy), \quad \frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=if'(x+iy).$$

Hence
$$i\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial x}=\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}.$$

On equating the real and imaginary parts, this gives

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}. \quad\ldots\ldots\ldots\ldots(22)$$

It can be shown conversely that if (22) holds, then $\phi+i\psi$ is a function of $x+iy$. For from (22) it follows that

$$i\left(\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}\right)=\left(\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}\right),$$

which is the condition that $\phi+i\psi$ has a differential coefficient with respect to $x+iy$.

It is evident that the families of curves given by $\phi=$ constant, $\psi=$ constant intersect orthogonally. Also, by differentiating (22) with respect to x and y and adding, it follows that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$

and by differentiating (22) with respect to y and x and subtracting, it follows that

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$$

The families of curves given by $\phi=$ constant and $\psi=$ constant are said to be conjugate. For (21) can be written in the form

$$\psi+i(-\phi)=-if(x+iy),$$

where ψ and $-\phi$ are respectively the real and imaginary parts of the function of a complex variable, $-if(x+iy)$, and the rôles of the two functions are now interchanged.

§ 47. **Solution of problems in two-dimensional steady irrotational motion.**

If we are given a special problem in two-dimensional motion and know that the motion is steady and irrotational, the straightforward method of procedure is to find a solution of the continuity equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

that satisfies the boundary conditions, and then determine p by means of the equation

$$\frac{p}{\rho}=-\Omega-\frac{1}{2}q^2+\mathrm{C},$$

$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2$ being substituted for q^2. If, however, we start with given boundary conditions, finding a solution may prove a very tedious or even impossible task.

The more fruitful method of procedure is not to take a particular problem and try to solve it, but to take a particular class of solutions of the differential equation and see to what problems they can be applied.

It has been shown in the last section that the real part ϕ of any function $f(x+iy)$ of a complex variable satisfies the equation for the velocity potential, and at the same time the imaginary part gives the stream function. We shall now take some simple functions of a complex variable and examine the solutions which we get in this way. Owing to the conjugate property of ϕ and ψ each solution will have a second meaning, *i.e.* we can also take ψ as potential and ϕ as stream function.

(1) $\phi+i\psi=x+iy.$

Here $\phi=x$ and $\psi=y$. The equipotential curves are straight lines parallel to Oy, and the stream lines are straight lines parallel to Ox.

(2) $\phi+i\psi=(x+iy)^2=x^2-y^2+2ixy.$

Here $\phi=x^2-y^2$ and $\psi=2xy$. The equipotential curves and the stream lines are represented in fig. 29. The equipotential curves are rectangular hyperbolas with the axes of coordinates as axes; the

stream lines are rectangular hyperbolas with the axes of coordinates as asymptotes.

In going from any curve of the family to the one next above it in the diagram, the parameter is increased by the same constant quantity. For example, it is the equipotential curves given by

$$x^2 - y^2 = 0, \quad x^2 - y^2 = 0{\cdot}2, \quad x^2 - y^2 = 0{\cdot}4, \quad \dots,$$

that are plotted. Hence the diagram represents the motion quantitatively as well as qualitatively. The quantity of liquid flowing per unit of time across the portion of any equipotential intercepted between any two consecutive stream lines is everywhere the same.

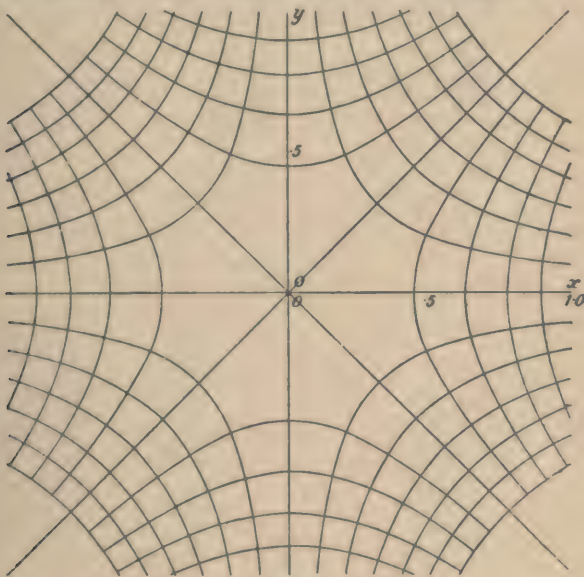

FIG. 29.

Any stream line may be taken as a fixed boundary. If, for example, we take the parts of the curves $xy = 0{\cdot}1$ and $xy = 0{\cdot}2$ in the first quadrant as boundaries, the solution represents the flow of a liquid in a channel with a bend in it.

If we take ϕ as the stream function and ψ as the potential function, we obtain the same solution turned through 45°.

$$(3) \quad \phi + i\psi = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

Therefore $$\phi = \frac{x}{x^2+y^2} \quad \text{or} \quad x^2+y^2-\frac{x}{\phi} = 0, \quad \dots\dots\dots\dots(23)$$

and $$\psi = -\frac{y}{x^2+y^2} \quad \text{or} \quad x^2+y^2+\frac{y}{\psi} = 0. \quad \dots\dots\dots\dots(24)$$

Equation (23) represents a family of circles with their centres on the x-axis and with the y-axis as common tangent at the origin. Equation (24) represents a family of circles with their centres on the y-axis and the x-axis as common tangent at the origin.

$$(4)\quad \phi + i\psi = \mu \log(x+iy) = \mu \log re^{i\theta} = \mu \log r + \mu i\theta.$$

Here $\phi = \mu \log r$ and $\psi = \mu\theta$. The equipotential lines are a system of concentric circles and the stream lines radii diverging from their common centre. The solution represents liquid either flowing from, or to the origin, or, as this fact is usually expressed, it represents either a two-dimensional source or sink at the origin.

If the rôles of potential function and stream function are interchanged, the solution represents liquid flowing round a circular cylinder. In this case the potential at any point is multiple-valued.

$$(5)\quad \phi + i\psi = \mu \log \frac{x+iy-a}{x+iy+a}.$$

Let the distances OA and OA′ in the figure each be equal to a and let P be the point $x+iy$. Then, if we write

$$x + iy - a = re^{i\theta},$$

$$r^2 = (x-a)^2 + y^2 \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x-a}.$$

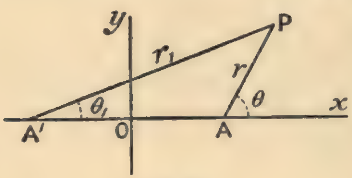

Fig. 30.

Hence $x+iy-a$ clearly represents the line AP and $x+iy+a$ represents the line A′P. Let

$$x + iy + a = r_1 e^{i\theta_1}.$$

Then
$$\phi + i\psi = \mu \log \frac{re^{i\theta}}{r_1 e^{i\theta_1}} = \mu \log \frac{r}{r_1} + \mu i(\theta - \theta_1).$$

Therefore $\phi = \mu \log r/r_1$ and $\psi = \mu(\theta - \theta_1)$. In order to examine the shape of these curves, let us change them into cartesians.

The first may be written

$$(x-a)^2 + y^2 = \mathrm{B}\{(x+a)^2 + y^2\},$$

where B is a constant. This gives

$$x^2 + y^2 + a^2 - 2ax\frac{1+\mathrm{B}}{1-\mathrm{B}} = 0. \quad \text{......................(25)}$$

The second may be written

$$\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x+a}=\text{const.} \quad \text{or} \quad \frac{\dfrac{y}{x-a}-\dfrac{y}{x+a}}{1+\dfrac{y^2}{x^2-a^2}}=\mathrm{C}.$$

This gives
$$x^2+y^2-a^2-\frac{2ay}{\mathrm{C}}=0. \qquad \text{.......................(26)}$$

Equation (25) represents a system of coaxal circles with the centres situated on the x-axis and the y-axis as radical axis. The radical axis intersects the system in imaginary points, and consequently the limiting points are real. Equation (26) represents a system of coaxal circles with the centres situated on the y-axis and the x-axis as radical axis. In this second system the radical axis cuts the curves in real points and the limiting points are imaginary.

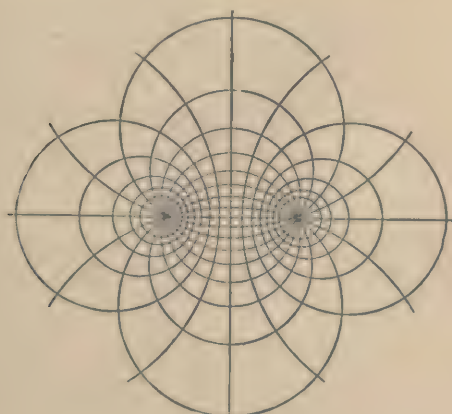

Fig. 31.

It is obvious from fig. 31 that the solution represents a source and a sink of equal intensities.

$$(6)\quad \phi+i\psi=\cosh^{-1}\frac{x+iy}{c}.$$

Here
$$x+iy=c\cosh(\phi+i\psi)=c\cosh\phi\cos\psi+ic\sinh\phi\sin\psi,$$

and on separating the real and imaginary parts, we obtain

$$x=c\cosh\phi\cos\psi, \quad y=c\sinh\phi\sin\psi.$$

On eliminating ψ, these equations give

$$\frac{x^2}{c^2\cosh^2\phi}+\frac{y^2}{c^2\sinh^2\phi}=1,$$

and on eliminating ϕ they give

$$\frac{x^2}{c^2\cos^2\psi}-\frac{y^2}{c^2\sin^2\psi}=1.$$

The equipotential curves are thus ellipses and the stream lines hyperbolas.

Both have common foci at $x=\pm c$. If the flat hyperbola, *i.e.* the curve for which $\sin\psi=0$, be taken as boundary, the solution represents

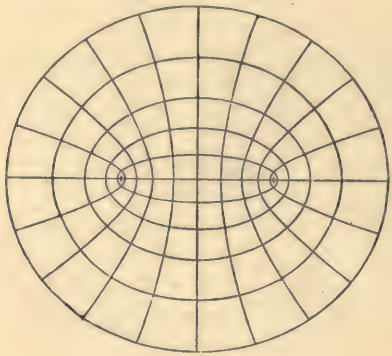

Fig. 32.

the flow of liquid through an aperture in a plane wall. If the rôles of ϕ and ψ be interchanged, the solution represents liquid circulating round an elliptic cylinder.

§ 48. **Application of the method of images.**

The method of images can also be applied to the solution of problems in hydrodynamics. For example, let AB be a fixed plane, the space to the right of which is filled with an incompressible liquid, let P be a fixed point source of strength m and let us suppose we are required to determine the motion. The liquid of course cannot penetrate through the plane.

In the case of a point source of strength m alone in an infinite liquid the velocity potential is m/r, where r is the distance of the point in question from the source, and the stream lines are obviously straight lines radiating from the source. The total quantity of liquid flowing from such a source in unit time is $4\pi m$.

The conditions to be satisfied in the given problem are that ϕ is zero at infinity, that the quantity of liquid flowing in unit time out of any surface in the fluid is zero unless the surface includes P, when it becomes $4\pi m$, and that the equipotential surfaces cut the plane AB at right angles. These conditions are satisfied by assuming that the source has an image of equal strength at P′, at an equal distance on

the other side of the plane AB. The potential at S is thus

$$\frac{m}{\mathrm{SP}}+\frac{m}{\mathrm{SP'}}.$$

An interesting result follows from this. Let S be on the plane at Q. Then the velocity at Q is $2m\mathrm{QN}/\mathrm{PQ}^3$ and has the direction QA. The pressure at Q, according to equation (18), is given by

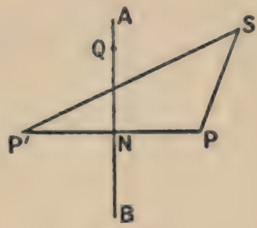

FIG. 33.

$$\frac{p}{\rho}=-\Omega-\frac{1}{2}q^2+\mathrm{C}.$$

Assume that there are no body forces, substitute for q^2 and we obtain

$$\frac{p}{\rho}=\mathrm{C}-2m^2\frac{\mathrm{QN}^2}{\mathrm{PQ}^6}.$$

The pressure on the plane is thus less than it would be if there were no source at P.

Writing $\mathrm{QN}=x$ and $\mathrm{PN}=a$, we find that the presence of the source diminishes the total thrust on the plane by

$$2\rho m^2\int_0^\infty\frac{x^2\,2\pi x\,dx}{(x^2+a^2)^3}=2\pi\rho m^2\int_0^\infty\left\{\frac{1}{(x^2+a^2)^2}-\frac{a^2}{(x^2+a^2)^3}\right\}d(x^2)$$

$$=2\pi\rho m^2\left(\frac{1}{a^2}-\frac{1}{2a^2}\right)=\frac{\pi\rho m^2}{a^2}.$$

EXAMPLES.

1. There is a line source and parallel to it a plane through which no liquid can pass. If the source and plane extend to infinity, find the velocity potential and the pressure on the plane.

2. Two impenetrable planes meet at right angles. One of the angles thus formed is filled with liquid, there being a continuous line source parallel to the line of intersection of the planes and distant respectively a and b from them. Derive an expression for the velocity potential and the pressure on the planes.

3. The motion of a liquid is in two dimensions, and there is a constant source at one point A in the liquid and an equal sink at another point B; find the form of the stream lines, and prove that the velocity at a point P varies as $(\mathrm{AP}\,.\,\mathrm{BP})^{-1}$, the plane of the motion being unlimited.

If the liquid is bounded by the planes $x=0$, $x=a$, $y=0$, $y=a$, and if the source is at the point $(0, a)$ and the sink at $(a, 0)$, find an expression for the velocity potential.

4. Liquid is moving irrotationally in two dimensions, between the space bounded by the two lines $\theta=\pm\frac{1}{6}\pi$ and the curve $r^3\cos 3\theta=a^3$. The bounding curves being at rest, prove that the velocity potential is of the form

$$\phi=r^3\sin 3\theta.$$

§ 49. **Motion of a sphere in an infinite liquid. No forces.**

A solid sphere of radius a and density σ is moving at a given instant with velocity v in a given direction in an incompressible liquid of density ρ. The liquid extends to infinity and is at rest there. It is required to investigate the motion of the sphere and the liquid.

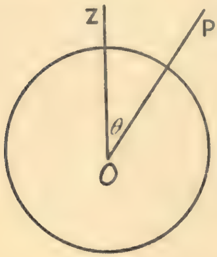

FIG. 34.

We shall assume that the motion is irrotational. Take the position of the centre of the sphere at a given instant as origin of coordinates and the direction of motion as the positive z-axis.

The velocity potential at a point P depends by symmetry only on OP and angle ZOP, *i.e.* on r and θ. Hence we obtain the equation of continuity in a suitable form by making the $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ terms in equation (7) equal to zero and then writing $\mathsf{U} = -\frac{\partial \phi}{\partial r}$, $\mathsf{V} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$. This gives

$$\left.\begin{aligned} &\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0 \\ \text{or} \qquad &\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0 \end{aligned}\right\}, \quad \text{................(27)}$$

the common factor ρ being cancelled out.

At infinity the velocity of the liquid is zero and at any point on the surface of the sphere the normal component is equal to the normal component at the point of the velocity of the surface. Hence the boundary conditions are

$$r = \infty, \quad \frac{\partial \phi}{\partial r} = 0, \quad r = a, \quad -\frac{\partial \phi}{\partial r} = v\cos\theta. \quad \text{..............(28)}$$

We determine ϕ completely by (27) and (28), and then obtain p by equation (17), *i.e.*

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2}q^2 + \mathsf{F}(t), \quad \text{..........................(29)}$$

Ω being put equal to zero, since there are no body forces acting on the liquid.

To solve (27) assume $\phi = f\cos\theta$, where f is a function of r only. This expression is suggested by the form of the boundary condition for $r = a$. Then, on substituting, the equation reduces to

$$r^2 \frac{\partial^2 f}{\partial r^2} + 2r\frac{\partial f}{\partial r} - 2f = 0.$$

In order to solve this, assume $f=r^m$. We find on substituting that $m=1$ or -2. We have therefore

$$\phi = \mathsf{A}r\cos\theta + \frac{\mathsf{B}}{r^2}\cos\theta,$$

where A and B are constants. Now

$$\frac{\partial\phi}{\partial r} = \mathsf{A}\cos\theta - \frac{2\mathsf{B}}{r^3}\cos\theta.$$

To satisfy the condition at infinity A must equal zero and to satisfy the condition at $r=a$ we must put

$$v = 2\mathsf{B}/a^3.$$

Hence
$$\phi = \frac{va^3}{2r^2}\cos\theta. \qquad \text{.............................(30)}$$

We shall now proceed to determine p, the pressure at P.

In order to find q, the velocity at P, we have

$$-\frac{\partial\phi}{\partial r} = \frac{va^3}{r^3}\cos\theta, \quad -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{va^3}{2r^3}\sin\theta.$$

Therefore
$$q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 = \left(\frac{va^3}{r^3}\right)^2\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right).$$

We cannot determine $\frac{\partial\phi}{\partial t}$ by differentiating ϕ with respect to t, for equation (30) gives only the value which ϕ takes for one particular value of t.

We know from symmetry that the sphere must move in a straight line. Let the position of its centre at time t be given by $(0, 0, \gamma)$; $\frac{d\gamma}{dt}=v$. Then the velocity potential is given by

$$\phi = \frac{va^3(z-\gamma)}{2\{x^2+y^2+(z-\gamma)^2\}^{\frac{3}{2}}},$$

since in this position
$$r^2 = x^2+y^2+(z-\gamma)^2$$

and
$$\cos\theta = \frac{z-\gamma}{\{x^2+y^2+(z-\gamma)^2\}^{\frac{1}{2}}}.$$

This gives

$$\frac{\partial\phi}{\partial t} = -\frac{va^3\dfrac{d\gamma}{dt}}{2\{x^2+y^2+(z-\gamma)^2\}^{\frac{3}{2}}} + \frac{3va^3(z-\gamma)^2\dfrac{d\gamma}{dt}}{2\{x^2+y^2+(z-\gamma)^2\}^{\frac{5}{2}}}$$
$$+\frac{a^3(z-\gamma)\dfrac{dv}{dt}}{2\{x^2+y^2+(z-\gamma)^2\}^{\frac{3}{2}}},$$

which becomes
$$\frac{\partial\phi}{\partial t} = -\frac{v^2a^3}{2r^3} + \frac{3v^2a^3\cos^2\theta}{2r^3} + \frac{a^3\cos\theta}{2r^2}\frac{dv}{dt},$$

on changing to our former notation.

On substituting the values of q^2 and $\frac{\partial\phi}{\partial t}$ in equation (29), we obtain

$$\frac{p}{\rho}=-\frac{v^2a^3}{2r^3}+\frac{3v^2a^3\cos^2\theta}{2r^3}+\frac{a^3\cos\theta}{2r^2}\frac{dv}{dt}-\frac{v^2a^6}{2r^6}\left(\cos^2\theta+\frac{1}{4}\sin^2\theta\right)+\mathrm{C}, \quad \ldots(31)$$

where C is a quantity depending only on the time. When the pressure is given at any definite point, C can be determined.

To find the kinetic energy of the liquid we may proceed by either of two ways. We may divide the fluid into elements, find the kinetic energy of each element, using the expression for q^2 already found, and integrate throughout the region occupied by the fluid. Or we may apply the theorem proved in § 45, namely, that

$$2\mathrm{T}=\rho\iint\phi\frac{\partial\phi}{\partial n}d\mathrm{S}.$$

In the given problem the liquid has two boundaries, one the sphere of radius a and another at infinity, which we may also take to be spherical and of radius R. Now for the boundary at infinity

$$\phi\frac{\partial\phi}{\partial n}=-\frac{va^3}{2r^2}\cos\theta\,.\,\frac{va^3}{r^3}\cos\theta,$$

and is of the order R^{-5}. The area of the boundary at infinity is of the order R^2. Hence the integral over the boundary at infinity is of the order R^{-3}, and vanishes when R is made very large. We obtain, therefore, for the integral over the whole boundary, taking $2\pi a^2\sin\theta\,d\theta$ as the element of area,

$$\rho\int_0^\pi\frac{(va^3)^2\cos^2\theta}{2a^5}2\pi a^2\sin\theta\,d\theta.$$

In the above expression $\frac{\partial}{\partial n}=-\frac{\partial}{\partial r}$, since the normal is drawn outwards from the liquid, *i.e.* inwards to the sphere. This gives

$$2\mathrm{T}=-\pi\rho a^3v^2\int_0^\pi\cos^2\theta\,d(\cos\theta)=\frac{2}{3}\pi\rho a^3v^2=\frac{1}{2}\mathrm{M}'v^2,$$

where M′ is the mass of the liquid displaced by the sphere. If M be the mass of the sphere, the total kinetic energy of the liquid and sphere is

$$\frac{1}{2}\left(\mathrm{M}+\frac{1}{2}\mathrm{M}'\right)v^2.$$

The effect of the liquid is thus equivalent to an addition to the inertia of the sphere of one half of the mass of the liquid displaced.

If the sphere is being accelerated,

$$\frac{1}{2}\mathrm{M}'v\frac{dv}{dt}$$

gives the rate at which the kinetic energy of the liquid is being increased. Consequently

$$\frac{1}{2}\mathsf{M}'\frac{dv}{dt}$$

must be the resultant force with which the sphere acts on the liquid, and

$$-\frac{1}{2}\mathsf{M}'\frac{dv}{dt}$$

the resultant force with which the liquid acts on the sphere. This force opposes the motion, and in magnitude and direction is quite independent of the previous motion of the sphere. If $\frac{dv}{dt}=0$, the liquid exerts no force on the sphere. Thus, if a sphere is set in motion in an ideal liquid and left to itself, it moves forward in a straight line with uniform velocity.

The resultant force exerted by the liquid on the sphere can also be derived from the expression for the pressure (31). For $r=a$, the latter becomes

$$\frac{p}{\rho}=\frac{v^2}{8}(9\cos^2\theta-5)+\frac{a\cos\theta}{2}\frac{dv}{dt}+\mathsf{C}.$$

Divide the sphere into elemental zones by planes parallel to the xy-plane. The area of a zone is $2\pi a^2\sin\theta\,d\theta$. From the form of p the resultant force must be in the direction of $\mathsf{O}z$. It is therefore given by

$$\int_0^\pi p\cos\theta\, 2\pi a^2\sin\theta\, d\theta.$$

Only the $\cos\theta$ term in the expression for p requires to be considered in the integration, because the other terms do not change with the sign of $\cos\theta$, *i.e.* they are the same both in front of and behind the sphere. The resultant force is therefore

$$\int_0^\pi \pi a^3\rho\frac{dv}{dt}\cos^2\theta\sin\theta\,d\theta=\pi a^3\rho\frac{dv}{dt}\int_{-1}^{+1}\cos^2\theta\,d(\cos\theta)$$
$$=\frac{2}{3}\pi a^3\rho\frac{dv}{dt}=\frac{1}{2}\mathsf{M}'\frac{dv}{dt},$$

the same result as before.

§ 50. **Motion of a sphere in an infinite liquid. Gravity acting.**

Take the direction of $\mathsf{O}z$ vertically downwards. The equation of continuity and the boundary conditions remain unaltered, and the velocity potential and kinetic energy are represented by the same expressions as before. Writing $-gz$ for Ω, the equation for p becomes

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}+gz-\frac{1}{2}q^2+\mathsf{F}(t).$$

The resultant thrust of the liquid on the sphere is in the direction of Oz, as before. The effect of the additional term is to give an additional force

$$\int_0^\pi g\rho a \cos^2\theta \, . \, 2\pi a^2 \sin\theta \, d\theta = 2\pi a^3 g\rho \int_{-1}^{+1} \cos^2\theta \, d(\cos\theta) = \frac{4}{3}\pi a^3 g\rho = \mathsf{M}'g,$$

acting vertically upwards. This is the force equal to the weight of liquid displaced, which is given by the principle of Archimedes.

If the sphere is allowed to fall under the action of gravity, the rate of increase of its momentum is

$$\frac{4}{3}\pi a^3 \sigma \frac{dv}{dt}.$$

The downward force on it due to gravity is

$$\frac{4}{3}\pi a^3 \sigma g.$$

The upward forces due to the weight of liquid displaced and to the communication of kinetic energy to the liquid are respectively

$$\frac{4}{3}\pi a^3 \rho g \quad \text{and} \quad \frac{2}{3}\pi a^3 \rho \frac{dv}{dt}.$$

The equation of motion is therefore

$$\frac{4}{3}\pi a^3 \sigma \frac{dv}{dt} = \frac{4}{3}\pi a^3 (\sigma - \rho) g - \frac{2}{3}\pi a^3 \rho \frac{dv}{dt},$$

which gives

$$\frac{dv}{dt} = \frac{g(\sigma - \rho)}{\sigma + \frac{\rho}{2}}$$

for the downward acceleration.

EXAMPLES.

1. Show that the lines of force of a small bar magnet are given by an equation similar to (30), and interpret the meaning of the constants in this case.

2. What difference does it make to the results of § 49 if we suppose the sphere fixed and the liquid flowing past it with velocity v?

3. Establish the special form of the equation of continuity suited to an incompressible fluid in which a right circular cylinder is moving with uniform velocity in a direction at right angles to its axis.

Find a solution of the equation for this case. Calculate the resultant velocity of the fluid at a point distant r from the axis and specify its direction. Find the kinetic energy of the fluid, and prove that if the density of the cylinder is equal to that of the fluid, the kinetic energy of the fluid motion is equal to that of the cylinder.

4. Discuss the characteristics of the motion for which

$$(1)\ \phi + i\psi = \frac{va^2}{(x+iy)}, \qquad (2)\ \phi + i\psi = v\left\{x + iy + \frac{a^2}{(x+iy)}\right\}.$$

(For solution, cf. Lamb's *Hydrodynamics*, p. 72.)

5. A large sphere of radius b is filled with liquid and a smaller sphere of radius a is moving inside it with velocity v along a diameter. Find the velocity potential and the kinetic energy of the liquid for the time when the spheres are concentric.

$$\left[\phi = \frac{a^3 v}{b^3 - a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta, \quad \mathsf{T} = \frac{1}{3}\pi a^3 \rho v^2 \frac{(b^3 + 2a^3)}{(b^3 - a^3)}.\right]$$

6. A cylinder of radius a is surrounded by a coaxal cylinder of radius b, and the intervening space is filled with liquid. The inner cylinder is moved with velocity u and the outer with velocity v along the same straight line perpendicular to the axis of the cylinders; prove that the velocity potential is

$$\phi = \frac{a^2 u - b^2 v}{b^2 - a^2} r\cos\theta + \frac{(u - v)a^2 b^2 \cos\theta}{(b^2 - a^2)r}.$$

§ 51. Let any curve be drawn from A to B. Take an element of length ds at a point P on the curve and let the direction of q, the resultant velocity at P, make an angle θ with ds. Then

$$\int_{\mathsf{A}}^{\mathsf{B}} q\cos\theta\, ds$$

is said to be the "flow" along the curve from A to B. Not the time aspect, only the instantaneous space aspect of the integral is meant.

Now
$$\cos\theta = \frac{u}{q}\frac{dx}{ds} + \frac{v}{q}\frac{dy}{ds} + \frac{w}{q}\frac{dz}{ds};$$

hence
$$\int_{\mathsf{A}}^{\mathsf{B}} q\cos\theta\, ds = \int_{\mathsf{A}}^{\mathsf{B}} (u\,dx + v\,dy + w\,dz),$$

and, if the motion is irrotational, this reduces to $\phi_{\mathsf{A}} - \phi_{\mathsf{B}}$.

If A and B coincide so that the curve becomes closed, then the integral

$$\int_{\mathsf{A}}^{\mathsf{A}} (u\,dx + v\,dy + w\,dz),$$

taken round the closed curve, is said to be the circulation round the closed curve or the circulation in the circuit. The circulation round any closed curve vanishes in a region in which a single-valued velocity potential exists. If the velocity potential is multiple-valued, the circulation does not necessarily vanish. For example, if $\phi = \mu\theta$ and the circuit goes once round the origin, the circulation is $2\mu\pi$.

§ 52. **Stokes' theorem.** The line integral of the tangential component of a vector taken round any closed curve is equal to the surface integral of the normal component of the curl of the same vector taken over any surface bounded by the curve, or

$$\int(u\,dx + v\,dy + w\,dz) = \iint\left\{l\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + m\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + n\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\} dS.$$

Let $P(x, y, z)$ be the centre of a rectangle ABCD, the lengths of the sides of which are dx, dy, and let u, v, w be the components of the vector at P.

At A and B u has respectively the values

$$u - \frac{\partial u}{\partial x}\frac{dx}{2} - \frac{\partial u}{\partial y}\frac{dy}{2}, \quad u + \frac{\partial u}{\partial x}\frac{dx}{2} - \frac{\partial u}{\partial y}\frac{dy}{2},$$

and at B and C v has respectively the values

$$v + \frac{\partial v}{\partial x}\frac{dx}{2} - \frac{\partial v}{\partial y}\frac{dy}{2}, \quad v + \frac{\partial v}{\partial x}\frac{dx}{2} + \frac{\partial v}{\partial y}\frac{dy}{2}.$$

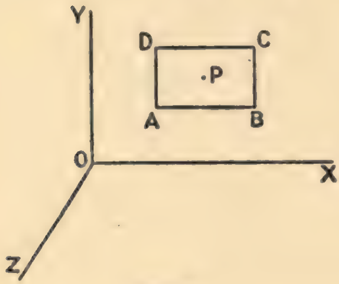

FIG. 35.

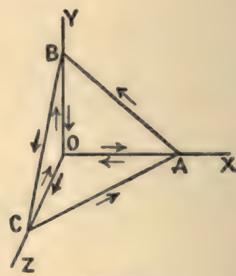

FIG. 36.

Hence the average value of u on AB is $u - \frac{\partial u}{\partial y}\frac{dy}{2}$ and the average value of v on BC is $v + \frac{\partial v}{\partial x}\frac{dx}{2}$. Similarly the average value of u on DC is $u + \frac{\partial u}{\partial y}\frac{dy}{2}$ and the average value of v on AD is $v - \frac{\partial v}{\partial x}\frac{dx}{2}$. The line integral round the element is therefore

$$\left(u - \frac{\partial u}{\partial y}\frac{dy}{2}\right)dx + \left(v + \frac{\partial v}{\partial x}\frac{dx}{2}\right)dy - \left(u + \frac{\partial u}{\partial y}\frac{dy}{2}\right)dx - \left(v - \frac{\partial v}{\partial x}\frac{dx}{2}\right)dy$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy.$$

Similar expressions hold when the rectangle is parallel to the YZ or ZX planes.

Now consider the triangular element ABC, the normal to which is given by l, m, n. Since the contributions from OA, OB, OC cut out, the line integral round ABC is obviously equal to the sum of the line integrals round ABO, BCO and CAO, that is to

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\triangle \text{ABO} + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\triangle \text{BCO} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\triangle \text{CAO}$$

by the result already proved. This becomes

$$\left\{l\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+m\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+n\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}\Delta,$$

where Δ is the area of ABC. Since the value of the circulation round ABC is independent of the coordinate system, this result holds for a triangle with its sides not parallel to the coordinate planes.

Take any surface and divide it up into elementary triangles. The line integral round the surface is equal to the sum of the line integrals round the individual triangles, because, as may be seen from fig. 37, every side of a triangle not at the same time on the bounding edge of the surface, is traversed twice during the integration in different directions, and so contributes nothing to the total. Hence the line integral round the surface is equal to

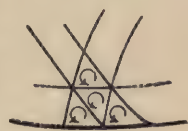

Fig. 37.

$$\int\left\{l\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+m\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+n\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}dS,$$

which proves the theorem.

§ 53. A word requires to be said about the direction of the normal to be considered positive. An observer walking round the edge of the surface on the positive side in the direction of the line integration would have the area on his left. This follows from fig. 35. The circulation is related to the positive direction of the normal in the manner typified by a right-handed screw.

It should be added that this rule is bound up with the convention adopted as to the coordinate axes. We always use in this book what is known as a right-handed or positive system, *i.e.* to an observer situated in succession at X, Y and Z the rotations in the directions YZ, ZX and XY are all anti-clockwise. The second figure below represents a left-handed or negative system; the corresponding rotations in it are clockwise. A reversal of the direction of any one axis changes a positive system into a negative one or vice versa. In defining the curl

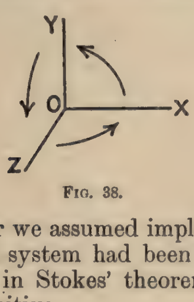

Fig. 38.

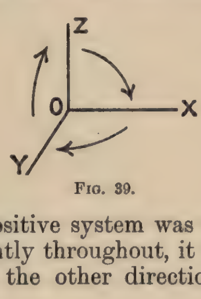

Fig. 39.

of a vector we assumed implicitly that a positive system was used. If a negative system had been used consistently throughout, it would be necessary in Stokes' theorem to consider the other direction of the normal positive.

§ 54. **Kelvin's circulation theorem.**

If the force is derivable from a potential and if the density is a function of the pressure only, the circulation in a circuit moving with the fluid does not alter with the time, that is

$$\frac{d}{dt}\int_{\mathsf{A}}^{\mathsf{A}}(u\,dx+v\,dy+w\,dz)=0.$$

We have $\quad \dfrac{d}{dt}(u\,dx)=\dfrac{du}{dt}\,dx+u\dfrac{d}{dt}(dx)=\left(-\dfrac{\partial\Omega}{\partial x}-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}\right)dx+u\,du.$

Hence

$$\frac{d}{dt}\int_{\mathsf{A}}^{\mathsf{A}}(u\,dx+v\,dy+w\,dz)=\int_{\mathsf{A}}^{\mathsf{A}}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\partial\Omega}{\partial x}\right)dx+\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\partial\Omega}{\partial y}\right)dy$$
$$+\left(-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\partial\Omega}{\partial z}\right)dz+u\,du+v\,dv+w\,dw$$
$$=\left(-\int\frac{dp}{\rho}-\Omega+\frac{1}{2}q^2\right)_{\mathsf{A}}^{\mathsf{A}}=0.$$

It should be noted that we have already met the same three quantities contained in the above bracket in Bernoulli's theorem, but then they had all the same sign.

It follows from this theorem that if any finite portion of a perfect fluid has a velocity potential at any instant, it has had one at all previous, and will have one at all subsequent times. For if the circulation is zero, the curl of the velocity is zero, and consequently the motion is irrotational.

§ 55. **Vortex tubes.**

A vortex line is a curve whose tangent at any point coincides with the direction at the point of the instantaneous axis of rotation of the element. It is thus the envelope of successive axes of rotation. Its equation is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}.$$

If vortex lines be drawn through every point on a small closed curve, the quantity of fluid enclosed is said to form a vortex or vortex tube or filament. The tube is taken so thin that the angular velocity is constant for all points on any one cross-section. Let ω be the angular velocity at any point on the tube and σ the cross-sectional area at that point; then $\omega\sigma$ is said to be the strength of the tube.

§ 56. **Laws of vortex motion.***

The following are the fundamental laws of vortex motion:

(1) A vortex filament is always composed of the same elements of fluid.

* These laws form the justification of the assumption of irrotational motion made in § 49.

(2) The strength of a vortex filament, $\omega\sigma$, is constant (a) with respect to time, (b) throughout the filament.

(3) Every vortex must either form a closed curve or have its extremities in the surface of the fluid.

Proof (1). Take any surface in the fluid wholly composed of vortex lines. By Stokes' theorem the circulation in any circuit in it is zero. After a time, owing to the motion of the fluid, the surface will have taken up a new position. By Kelvin's theorem the circulation in any circuit in the surface is still zero; hence the surface is still composed of vortex lines.

If two such surfaces be considered, their intersection must always be a vortex line. Hence vortex lines move with the fluid.

($2a$) This follows from Kelvin's theorem, since by Stokes' theorem $2\omega\sigma$ is the circulation in a circuit round the tube.

($2b$) *Helmholtz's proof*: Isolating in imagination a portion of a vortex bounded by two normal sections and applying Gauss's theorem to it, we have

$$\iint (l\xi + m\eta + n\zeta)\, d\mathsf{S} = \iiint \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) dx\, dy\, dz.$$

The integrand on the right-hand side vanishes throughout the volume, as may be found by actual differentiation of ξ, η and ζ, and the integrand on the left-hand side vanishes over the surface of the filament, since the normal component of the angular velocity is zero there. We are left therefore with only the surface integral over the ends of the filament, and consequently it must be zero. It has the value

$$\omega_1\sigma_1 - \omega_2\sigma_2,$$

FIG. 40.

ω_1, ω_2, σ_1, σ_2 being respectively the angular velocities and cross-sectional areas at the ends of the element. Hence the theorem follows. The minus sign is accounted for by the fact that the angular velocity has the same direction and the normal different directions at the two ends.

($2b$) *Kelvin's proof*: Apply Stokes' theorem to the portion of the surface of the tube bounded by ABCDEFG. Since $\iint (l\xi + m\eta + n\zeta)\, d\mathsf{S} = 0$ for this surface, $\int (u\, dx + v\, dy + w\, dz)$ taken along the boundary is also equal to zero. Now if BC and GF are taken sufficiently close together, the part of the line integral along BC is equal and opposite to the part of the line integral along FG. The whole line integral thus reduces to the parts round the rings GAB and CDEF. These are equal respectively to $2\omega_1\sigma_1, -2\omega_2\sigma_2$ in the notation of the preceding proof. Hence the theorem.

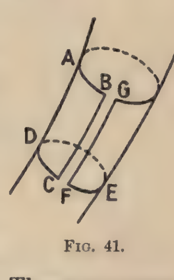

FIG. 41.

(3) If a vortex tube ended in a fluid a closed surface could be drawn cutting the vortex only once, and $\iint (l\xi + m\eta + n\zeta)dS$ taken over this surface would not be zero.

§ 57. **The rectilinear vortex.**

Let the motion be in two dimensions. Then $w=0$, $\frac{\partial u}{\partial z}=0$, $\frac{\partial v}{\partial z}=0$, and consequently $\xi=0$, $\eta=0$. If vortex lines exist, they must be parallel to the z-axis. In two-dimensional motion we can always write $u=-\frac{\partial\psi}{\partial y}$, $v=\frac{\partial\psi}{\partial x}$. Consequently

$$2\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}.$$

Let us suppose that we have a vortex of circular cross-section at rest in an infinite liquid which is itself at rest at infinity. Let the radius of the cross-section be a, let the centre of the vortex be situated at the origin and let the angular velocity have the constant value ζ throughout the vortex. Then ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2\zeta$$

inside the vortex and the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

outside the vortex. As in this problem we obviously have symmetry about the axis of the vortex, it is better to take r as independent variable. The best way of changing to the new independent variable is in this case by means of the formula (cf. § 24)

$$\nabla^2 V=\frac{1}{\lambda\mu\nu}\Sigma\frac{\partial}{\partial\xi}\left(\frac{\mu\nu}{\lambda}\frac{\partial V}{\partial\xi}\right).$$

On retaining only the differentiation with respect to ξ and substituting $\xi=r$, $\lambda=\nu=1$, $\mu=r$, the equations become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=2\zeta \quad\text{..............................(32)}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=0. \quad\text{..............................(33)}$$

The radial and tangential components of the velocity are now given respectively by $-\frac{1}{r}\frac{\partial\psi}{\partial\theta}$ and $\frac{\partial\psi}{\partial r}$.

Solving (33), we have

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=0, \quad r\frac{\partial\psi}{\partial r}=\mathsf{C} \quad \text{and} \quad \psi=\mathsf{C}\log r+\mathsf{D},$$

and solving (32), we obtain

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=2\zeta r, \quad r\frac{\partial\psi}{\partial r}=\zeta r^2+\mathsf{A}, \quad \psi=\frac{1}{2}\zeta r^2+\mathsf{A}\log r+\mathsf{B}.$$

The radial component of the velocity, it will be observed, is everywhere zero.

We have now to determine the constants of integration. If we measure ψ from the origin it must be zero when $r=0$. Hence, in the expression for ψ inside the vortex, $\mathsf{A}=\mathsf{B}=0$. At the surface of the vortex there must be no slipping; the tangential component of the velocity must be the same there both inside and outside. The value inside is ζa and the value outside C/a; hence $\mathsf{C}=\zeta a^2$. Also, since the two expressions for ψ must agree for $r=a$, $\frac{1}{2}\zeta a^2=\zeta a^2\log a+\mathsf{D}$. Hence $\mathsf{D}=\frac{1}{2}\zeta a^2-\zeta a^2\log a$. Substitute m/π for ζa^2, m being the strength of the vortex. We then have

$$\psi=\frac{mr^2}{2\pi a^2}$$

inside the vortex and

$$\psi=\frac{m}{\pi}\log r/a+\frac{m}{2\pi}$$

outside the vortex.

These expressions are of the same form as the expressions for the gravitational potential due to an infinitely long circular cylinder inside and outside the cylinder. To obtain the latter (cf. § 19) all we have to do is to substitute $-k$ for $\frac{1}{2\pi}$.

As has been mentioned above, the direction of the resultant velocity at any point is tangential to the circle drawn through the point with its centre on the axis. Its value inside is given by

$$\frac{\partial}{\partial r}\frac{mr^2}{2\pi a^2}=\frac{mr}{\pi a^2}$$

and its value outside by

$$\frac{\partial}{\partial r}\frac{m}{\pi}\log r/a=\frac{m}{\pi r}.$$

It thus agrees (cf. § 116) both in magnitude and direction with the magnetic intensity due to an electric current of strength $m/2\pi$ electromagnetic units flowing along a homogeneous conductor coincident with the vortex.

We can determine the pressure outside by the equation

$$\frac{p}{\rho}+\frac{1}{2}q^2=\frac{\partial\phi}{\partial t}+\text{C},$$

putting $\dfrac{\partial\phi}{\partial t}=0$ and $q^2=\dfrac{m^2}{\pi^2r^2}$. Then

$$\frac{p}{\rho}=\frac{\Pi}{\rho}-\frac{m^2}{2\pi^2r^2},$$

Π being the pressure at infinity.

In order to find the pressure inside the vortex, write down the equation of motion for the radial direction. The body force is zero. The acceleration is $\dfrac{m^2r}{\pi^2a^4}$. Hence we have

$$\frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{m^2r}{\pi^2a^4}.$$

This gives $\dfrac{p}{\rho}=\dfrac{m^2r^2}{2\pi^2a^4}+\dfrac{\text{P}}{\rho}$, where P is the pressure at the centre of the vortex.

At the surface of the vortex

$$\frac{\Pi}{\rho}-\frac{m^2}{2\pi^2a^2}=\frac{m^2}{2\pi^2a^2}+\frac{\text{P}}{\rho},\quad\text{therefore}\quad\frac{\text{P}}{\rho}=\frac{\Pi}{\rho}-\frac{m^2}{\pi^2a^2}.$$

The pressure diminishes all the way from infinity to the centre. If $\Pi<m^2\rho/(\pi^2a^2)$, p becomes negative for some value of $r<a$, and in this case a cylindrical hollow exists in the vortex. It is even possible for the vortex to be all hollow.

§ 58. If a liquid of invariable density is moving irrotationally, its kinetic energy (cf. § 45) is given by

$$\text{T}=\frac{1}{2}\rho\iiint q^2\,dx\,dy\,dz=\frac{1}{2}\rho\iint\phi\frac{\partial\phi}{\partial n}\,d\text{S}.$$

Let the boundary be fixed. Then $\dfrac{\partial\phi}{\partial n}=0$, and consequently $\text{T}=0$.

But every element of the volume integral is positive. Hence $q^2=0$, *i.e.* we cannot have liquid moving irrotationally inside a space with fixed boundaries.

§ 59. If an impulsive pressure acts on a liquid at rest producing a velocity u, v, w,

$$u=-\frac{1}{\rho}\frac{\partial\omega}{\partial x},\ \ldots,\ \ldots,$$

(cf. § 37). Let the liquid be incompressible and put $\phi=\omega/\rho$. Then $u=-\dfrac{\partial\phi}{\partial x}$, …, …, *i.e.* the motion produced by impulsive pressure in a liquid at rest is irrotational.

Conversely, any irrotational motion existing may be imagined produced by impulsive pressure.

§ 60. **Kelvin's minimum energy theorem.**

The irrotational motion of a liquid occupying a simply-connected region has less kinetic energy than any other motion consistent with the same motion of the boundary. (A region is said to be simply-connected when every closed surface drawn in it can be contracted to a point without passing out of the region.)

Let $-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial z}$ give the irrotational motion. Let $-\frac{\partial\phi}{\partial x}+u'$, $-\frac{\partial\phi}{\partial y}+v'$, $-\frac{\partial\phi}{\partial z}+w'$ give another possible motion. The motion of the boundary is the same in both cases; hence, at a point on the boundary where the normal is given by l, m, n, $lu'+mv'+nw'=0$.

The kinetic energy of the other possible motion is given by

$$\mathrm{T}=\frac{1}{2}\rho\iiint\Sigma\left(-\frac{\partial\phi}{\partial x}+u'\right)^2 dx\,dy\,dz$$

$$=\frac{1}{2}\rho\iiint\Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 dx\,dy\,dz+\frac{1}{2}\rho\iiint\Sigma u'^2\,dx\,dy\,dz-\rho\iiint\Sigma\frac{\partial\phi}{\partial x}u'\,dx\,dy\,dz.$$

But by Green's theorem, since the liquid is incompressible,

$$\iiint\Sigma\frac{\partial\phi}{\partial x}u'\,dx\,dy\,dz=-\iiint\phi\Sigma\frac{\partial u'}{\partial x}\,dx\,dy\,dz+\iint\phi\,(lu'+mv'+nw')\,d\mathrm{S}=0,$$

and $\iiint\Sigma u'^2\,dx\,dy\,dz$ is essentially positive. Hence the theorem is proved.

The theorem is a particular case of a more general theorem due to Kelvin, which is enunciated as follows:

A material system if started from rest by impulses applied to certain points, adjusted to communicate certain specified velocities to these points, has less kinetic energy than any other possible motion of the system fulfilling the same velocity conditions.

EXAMPLES.

1. Considering the earth as composed of a solid spherical part, of density symmetrical about the centre, covered by a stratum of water, and disregarding the attractions of the particles of water on one another, prove that the equation of the free surface is

$$\frac{\mu}{\sqrt{x^2+y^2+z^2}}+\frac{1}{2}\omega^2(x^2+y^2)=\frac{\mu}{b},$$

where x, y, z are the coordinates of the point considered, taken from the centre as origin (z being taken along the axis), ω is the angular velocity of rotation, b is the polar radius, and μ is a constant to be interpreted.

Show that if a be the equatorial radius,

$$\frac{a-b}{a}=\frac{1}{2}\omega^2\frac{a^2b}{\mu}.$$

Work out this quantity numerically to a rough approximation. (Hint: The flow along an arc of a meridian terminating at a pole is constant.)

2. An ellipsoidal hollow space (equation $x^2/a^2+y^2/b^2+z^2/c^2=1$) is filled with water. The water has vorticity, uniform throughout the mass, of angular speeds ξ, η, ζ about the principal axes of the surface, and the case is turning with angular speeds ω_1, ω_2, ω_3 about the same axes. Show by the condition that the motion must fulfil at the surface of the vessel that the velocity potential of the irrotational part of the motion of the water at any point of coordinates x, y, z referred to the principal axes is

$$\phi=-(\omega_1-\xi)\frac{b^2-c^2}{b^2+c^2}yz-(\omega_2-\eta)\frac{c^2-a^2}{c^2+a^2}zx-(\omega_3-\zeta)\frac{a^2-b^2}{a^2+b^2}xy.$$

Prove that the component speeds with reference to fixed axes with which those of the ellipsoid coincide at the instant are

$$u=z\eta-y\zeta+(\omega_2-\eta)\frac{c^2-a^2}{c^2+a^2}z+(\omega_3-\zeta)\frac{a^2-b^2}{a^2+b^2}y,$$

with similar expressions for v, w.

Hence find the component speeds u', v', w' of the water at x, y, z relative to the moving axes of the ellipsoid, and prove that a given particle remains on an ellipsoidal surface similar to the containing surface.

3. If the axis of a hollow vortex be the axis of z, measured vertically downwards, the plane of xy being the asymptotic plane to the free surface, and if Π be the atmospheric pressure, prove that the equation of the surface, at which the pressure is $\Pi+g\rho a$, is

$$(x^2+y^2)(z-a)=c^3,$$

where c is a constant.

CHAPTER III.

FOURIER SERIES AND CONDUCTION OF HEAT.

§ 61. SUPPOSE that we are given a curve $y=f(x)$. Then in the equation

$$y=a_0+a_1\cos x+b_1\sin x,$$

a_0, a_1 and b_1 may be determined so that the graph of the equation cuts the curve in any three points between $x=0$ and $x=2\pi$. For it is only necessary to write down the condition that the ordinates should be equal at these three points and we have three linear equations for determining a_0, a_1 and b_1. Similarly, in the equation

$$\begin{aligned} y=a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\ldots\, a_n\cos nx \\ +b_1\sin x+b_2\sin 2x+b_3\sin 3x+\ldots\, b_n\sin nx. \end{aligned}$$

$a_0, a_1, a_2, a_3 \ldots a_n, b_1, b_2, b_3 \ldots b_n$ may be determined so that the two curves cut in $(2n+1)$ points. If n be made infinitely great, the two curves will cut in an infinite number of points. This raises the question whether the curves will not touch throughout the range $x=0$ to $x=2\pi$, whether it is not possible to represent any function throughout the range by an infinite series of the above type.

§ 62. Let us assume the possibility of expanding $f(x)$ throughout the range 0 to 2π in a series of the above type, *i.e.* in a Fourier series, and let us assume that the series may be integrated term by term. Then

$$f(x)=a_0+a_1\cos x+a_2\cos 2x\ldots+b_1\sin x+b_2\sin 2x. \quad \text{.........(1)}$$

Integrate both sides of the equation with respect to x from 0 to 2π. On the left-hand side we have $\int f(x)\,dx$. On the right-hand every term disappears except the first, which gives $2\pi a_0$. Hence

$$a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx. \quad \text{..............................(2)}$$

Multiply both sides of (1) by $\cos nx$, where n is any integer, and integrate with regard to x from 0 to 2π. Then, on the right-hand side, we have terms of the following type:

$$a_0\int \cos nx\,dx,$$

$$a_n\int \cos^2 nx\,dx=\frac{a_n}{2}\int(1+\cos 2nx)\,dx,$$

$$a_m\int\cos mx\cos nx\,dx=\frac{a_m}{2}\int\{\cos(m+n)x+\cos(m-n)x\}\,dx,$$

$$b_n\int\sin nx\cos nx\,dx=\frac{b_n}{2}\int\sin 2nx\,dx,$$

$$b_m\int\sin mx\cos nx\,dx=\frac{b_m}{2}\int\{\sin(m+n)x+\sin(m-n)x\}\,dx,$$

m being any integer except n. It is clear, that on integrating and substituting the limits 0 and 2π, every one of these terms will vanish except the second. It gives $a_n\pi$. The left-hand gives

$$\int_0^{2\pi}f(x)\cos nx\,dx.$$

Hence $$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx. \qquad\qquad (3)$$

Similarly, by multiplying both sides of (1) by $\sin nx$ and integrating between the same limits, it can be shown that

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx.$$

The two formulae (2) and (3) may be combined into one by writing the absolute term in the series $\frac{a_0}{2}$ instead of a_0. Here, however, it will always be written a_0.

If the range is taken from $-\pi$ to $+\pi$ instead of from 0 to 2π, the only difference in the formulae for the coefficients is that the limits of integration are from $-\pi$ to $+\pi$.

§ 63. It is to be noted that in the preceding section we assumed, but did not prove, that the expansion of $f(x)$ in a series of the required type was possible. It may be shown by trial, *i.e.* by taking particular cases, calculating the coefficients and comparing the numerical value of the function with the sum of the first few terms of the series, that the assumptions are justified. This experimental method of proving the assumptions is more convincing from the student's point of view than the rigorous proof which is due to Dirichlet. Dirichlet's treatment of the subject is long and will not be given here. In it the sum of n terms of the series is taken, and it is shown that when n becomes infinitely great, the sum approaches $f(x)$, *provided that $f(x)$ is single-valued and finite and has only a finite number of discontinuities and turning-values from $x=0$ to $x=2\pi$.* This gives the condition on which the expansion of $f(x)$ in a Fourier series is possible. If there are no discontinuities in $f(x)$, the series is equal to $f(x)$ between 0 and 2π. At a discontinuity in $f(x)$ the value of the series is the mean of the values of $f(x)$ on both sides of the discontinuity. At 0 and 2π

the value of the series is the mean of the values of $f(x)$ at these two points.

It is not necessary that $f(x)$ should have the same mathematical expression throughout the range. For example, it may consist of several different and disconnected straight lines.

A Fourier series can always be integrated term by term, but cannot in general be differentiated term by term. It is easy to see why this should be. For, if we differentiate the right-hand side of

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \ldots,$$

we obtain

$$- a_1 \sin x - 2a_2 \sin 2x - 3a_3 \sin 3x \ldots$$
$$+ b_1 \cos x + 2b_2 \cos 2x + 3b_3 \cos 3x \ldots.$$

A Fourier series converges only because the coefficients of successive terms decrease. It is obvious that differentiation must either destroy this convergence or make it less rapid.

On integrating the same series, we obtain

$$a_0 x + a_1 \sin x + \frac{a_2}{2} \sin 2x + \frac{a_3}{3} \sin 3x \ldots$$
$$- b_1 \cos x - \frac{b_2}{2} \cos 2x - \frac{b_3}{3} \cos 3x \ldots,$$

and it is obvious that integration increases the convergence.

Examples. (1) Let $f(x) = x^2$ from $x=0$ to $x=2\pi$.

Then $a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{4}{3}\pi^2,$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$$
$$= \left(\frac{1}{n\pi} x^2 \sin nx + \frac{2}{n^2\pi} x \cos nx\right)_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \cos nx\, dx = \frac{4}{n^2}$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx$

$$= \left(-\frac{1}{n\pi} x^2 \cos nx + \frac{2}{n^2\pi} x \sin nx\right)_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \sin nx\, dx = -\frac{4\pi}{n}.$$

Hence, from 0 to 2π,

$$x^2 = \frac{4}{3}\pi^2 + \Sigma \frac{4}{n^2} \cos nx - \Sigma \frac{4\pi}{n} \sin nx.$$

(2) Let $f(x)=x$ from $x=0$ to $x=\pi$ and $f(x)=x-2\pi$ from $x=\pi$ to $x=2\pi$ (fig. 42).

Here $$a_0=\frac{1}{2\pi}\left\{\int_0^\pi x\,dx+\int_\pi^{2\pi}(x-2\pi)\,dx\right\}=0,$$

$$a_n=\frac{1}{\pi}\left\{\int_0^\pi x\cos nx\,dx+\int_\pi^{2\pi}(x-2\pi)\cos nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\int_0^{2\pi}x\cos nx\,dx-2\pi\int_\pi^{2\pi}\cos nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left(\frac{x}{n}\sin nx\right)_0^{2\pi}-\int_0^{2\pi}\frac{\sin nx}{n}\,dx-\left(\frac{2\pi}{n}\sin nx\right)_\pi^{2\pi}\right\}=0$$

and $$b_n=\frac{1}{\pi}\left\{\int_0^\pi x\sin nx\,dx+\int_\pi^{2\pi}(x-2\pi)\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\int_0^{2\pi}x\sin nx\,dx-2\pi\int_\pi^{2\pi}\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left(-\frac{x}{n}\cos nx\right)_0^{2\pi}+\int_0^{2\pi}\frac{\cos nx}{n}\,dx+\frac{2\pi}{n}\left(\cos nx\right)_\pi^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left\{-\frac{2\pi}{n}\cos 2n\pi+\frac{2\pi}{n}\cos 2n\pi-\frac{2\pi}{n}\cos n\pi\right\}=\frac{2}{n}(-1)^{n+1}.$$

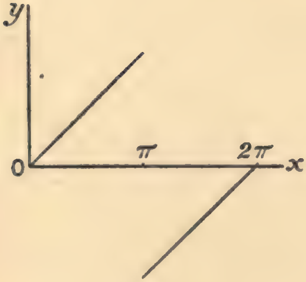

FIG. 42.

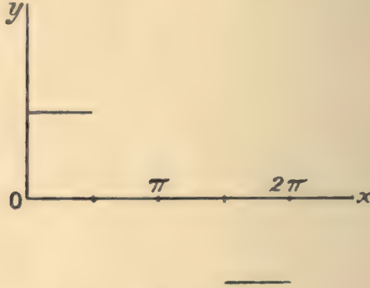

FIG. 43.

Hence the series is

$$2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x\ldots\right].$$

At $x=\pi$ there is a discontinuity in $f(x)$, the value changing from π to $-\pi$. On substituting the value $x=\pi$ in the series, we find, as was to be expected, that it gives the mean of π and $-\pi$, namely zero.

(3) Let (fig. 43) $f(x) = \quad c$ from $x=0$ to $x=\frac{\pi}{2}$,

$$= \quad 0 \text{ from } x=\frac{\pi}{2} \text{ to } x=\frac{3\pi}{2}$$

and $$= -c \text{ from } x=\frac{3\pi}{2} \text{ to } x=2\pi.$$

Then $$a_0=\frac{1}{2\pi}\left\{\int_0^{\frac{\pi}{2}} c\,dx - \int_{\frac{3\pi}{2}}^{2\pi} c\,dx\right\}=0,$$

$$a_n=\frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} c\cos nx\,dx - \int_{\frac{3\pi}{2}}^{2\pi} c\cos nx\,dx\right\}$$

$$=\frac{c}{n\pi}\left\{\sin\frac{n\pi}{2} - \sin 2n\pi + \sin\frac{3n\pi}{2}\right\}=0$$

and $$b_n=\frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} c\sin nx\,dx - \int_{\frac{3\pi}{2}}^{2\pi} c\sin nx\,dx\right\}$$

$$=\frac{c}{n\pi}\left\{1-\cos\frac{n\pi}{2}+\cos 2n\pi - \cos\frac{3n\pi}{2}\right\}$$

$$=\frac{c}{n\pi}\left\{2-2\cos n\pi\cos\frac{n\pi}{2}\right\}$$

$=$ alternately $\dfrac{4c}{n\pi}$ and 0 if n be even, $\dfrac{2c}{n\pi}$ if n be odd.

Hence the series is

$$\frac{2c}{\pi}\left[\sin x + \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \ldots\right].$$

It is obvious from inspection that when x is 0 or 2π the series is zero, that is, the mean of the values which $f(x)$ has for these two points.

§ 64. Other forms of Fourier's series.

We had $f(x)=a_0+a_1\cos x+a_2\cos 2x+\ldots+b_1\sin x+b_2\sin 2x\ldots$ throughout the range $x=0$ to $x=2\pi$, the coefficients being given by

$$a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

and $$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

It is sometimes necessary to represent $f(x)$ by a series throughout the more general range $x=0$ to $x=2l$. Write $x=\frac{\pi z}{l}$. Then, when $x=2\pi$, $z=2l$, and (1) may be written

$$f\left(\frac{\pi z}{l}\right)=\phi(z)=a_0+a_1\cos\frac{\pi z}{l}+a_2\cos\frac{2\pi z}{l}\dots+b_1\sin\frac{\pi z}{l}+b_2\sin\frac{2\pi z}{l}\dots,$$

where $$a_0=\frac{1}{2l}\int_0^{2l}\phi(z)\,dz,\quad a_n=\frac{1}{l}\int_0^{2l}\phi(z)\cos\frac{n\pi z}{l}\,dz$$

and $$b_n=\frac{1}{l}\int_0^{2l}\phi(z)\sin\frac{n\pi z}{l}\,dz.$$

Substituting now x for z and $f(x)$ for $\phi(z)$, we find that throughout the range $x=0$ to $x=2l$.

$$f(x)=a_0+a_1\cos\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}\dots+b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}\dots,\quad\dots(4)$$

where $$a_0=\frac{1}{2l}\int_0^{2l}f(x)\,dx,\quad a_n=\frac{1}{l}\int_0^{2l}f(x)\cos\frac{n\pi x}{l}\,dx$$

and $$b_n=\frac{1}{l}\int_0^{2l}f(x)\sin\frac{n\pi x}{l}\,dx.$$

Again, $$\cos\frac{n\pi x}{l}=\cos\frac{n\pi(2l-x)}{l}.$$

The graph of the series

$$a_0+a_1\cos\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}\dots$$

is therefore symmetrical about the ordinate through the middle of its range, *i.e.* the ordinates on opposite sides of $x=l$ at the same distance from it are equal and have the same sign. Also

$$\sin\frac{n\pi x}{l}=-\sin\frac{n\pi(2l-x)}{l}.$$

The graph of the series

$$b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}\dots$$

is therefore symmetrical about the middle point of its range, *i.e.* the ordinates on opposite sides of the point $x=l$, $y=0$ at equal distances from it are equal in magnitude but opposite in sign. In the Fourier series representing a function of x symmetrical about the middle ordinate, therefore, there will be no sine terms, and in the series representing a function symmetrical about the middle point, there will be no cosine and no absolute terms.* In examples 2 and 3 of the preceding section the functions are symmetrical about the middle point, and we found on evaluating the coefficients, that the absolute term and cosine terms

* If the origin of coordinates is at the middle of the range, these functions become respectively even and odd.

vanished; this result might, however, have been inferred from the nature of the function.

Suppose that $f(x)$ is given from $x=0$ to $x=l$. Then we may expand it in a series in three different ways. We may first of all suppose that l is only half the range and fill in the function in the second half so as to make it symmetrical about the ordinate through $x=l$. Then

$$f(x)=a_0+a_1\cos\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}\ldots, \quad \ldots\ldots\ldots\ldots(5)$$

where $$a_0=\frac{1}{2l}\int_0^{2l}f(x)\,dx=\frac{1}{l}\int_0^{l}f(x)\,dx$$

and $$a_n=\frac{1}{l}\int_0^{2l}f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^{l}f(x)\cos\frac{n\pi x}{l}\,dx.$$

The second formulae follow since the integrals have the same values in the first and second halves of the range. The above expansion is called the half range cosine series.

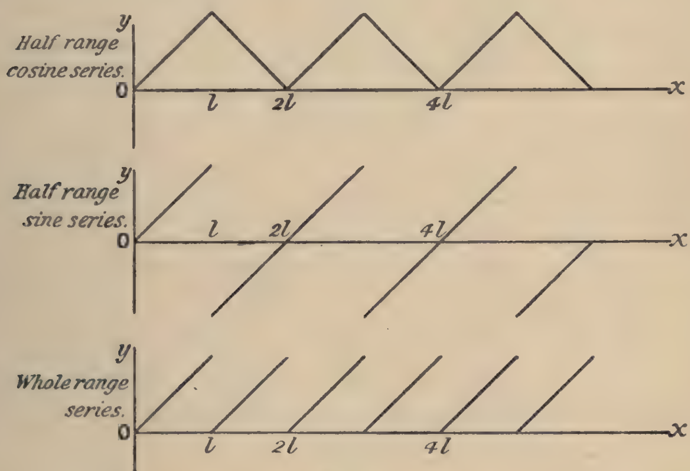

FIG. 44.

We may suppose that l is only half the range and fill in the function in the second half so as to make it symmetrical about the point $x=l$, $y=0$. Then

$$f(x)=b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}+b_3\sin\frac{3\pi x}{l}\ldots, \quad \ldots\ldots\ldots\ldots(6)$$

where $$b_n=\frac{1}{l}\int_0^{2l}f(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^{l}f(x)\sin\frac{n\pi x}{l}\,dx.$$

This expansion is called the half range sine series. The second formula for b_n follows since the integral has the same value in the

first and second halves of the range. It is of course always the second formulae that are used in deriving the coefficients in the half range series.

Finally, we may suppose that l is the whole range and use (4), substituting l for $2l$.

The difference between the different methods is made much clearer by consideration of a simple case. Suppose that $f(x)=x$ from 0 to l. Then the three diagrams on p. 75 represent the three series as functions of x.

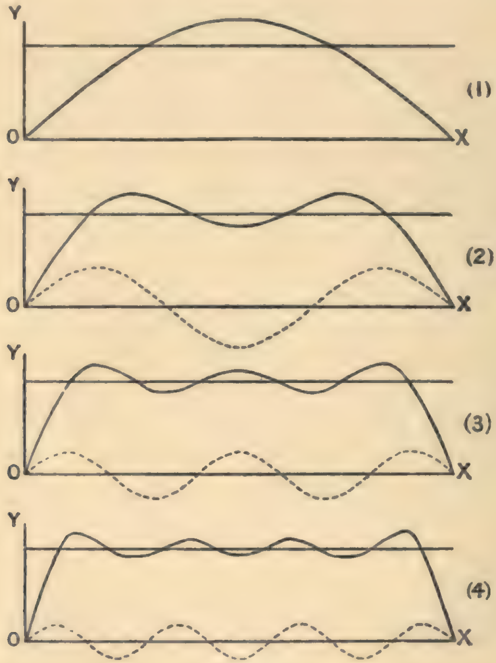

FIG. 45.

§ 65. It is instructive and interesting to plot the first few terms in a trigonometrical series as curves, and to show how their sum approaches the value of the function. For example, let $f(x)=\frac{\pi}{4}$ from $x=0$ to $x=\pi$. Then, if we use the half range sine series, substituting π for l in equation (6),

$$b_n=\frac{2}{\pi}\int_0^\pi \frac{\pi}{4}\sin nx\,dx = -\frac{1}{2n}(\cos n\pi - 1)$$

$$=\frac{1}{n} \text{ if } n \text{ is odd, } 0 \text{ if } n \text{ is even.}$$

Hence $$\frac{\pi}{4}=\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x \dots .$$

In the accompanying set of figures, the horizontal line represents $y=\frac{\pi}{4}$. The heavy curve in (1) represents the first term of the series. In (2) the dotted curve gives the second term of the series and the heavy curve the sum of the first two terms. In (3) the dotted curve gives the third term of the series and the heavy curve the sum of the first three terms. In (4) the dotted curve gives the fourth term of the series and the heavy curve the sum of the first four terms. We see from the figures how the sum of the terms gradually approximates to a straight line.

EXAMPLES.

1. Expand $f(x)=x$ from 0 to $\frac{\pi}{2}$ and $=\pi-x$ from $\frac{\pi}{2}$ to π as a half range sine series.

Result: $$\frac{4}{\pi}\left(\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}\dots\right).$$

2. Show by expansion in a half range sine series that
$$x^2=\frac{2}{\pi}\left[\left(\frac{\pi^2}{1}-\frac{4}{1^3}\right)\sin x-\frac{\pi^2}{2}\sin 2x+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin 3x\right.$$
$$\left.-\frac{\pi^2}{4}\sin 4x+\left(\frac{\pi^2}{5}-\frac{4}{5^3}\right)\sin 5x\dots\right].$$

3. Expand $f(x)=x\sin x$ from 0 to π as a half range cosine series.

Result: $$1-\frac{\cos x}{2}-\frac{2\cos 2x}{1.3}+\frac{2\cos 3x}{2.4}-\frac{2\cos 4x}{3.5}\dots .$$

4. In the interval $0<x<\frac{l}{2}$, $f(x)=\frac{1}{4}l-x$, and in the interval $\frac{l}{2}<x<l$, $f(x)=x-\frac{3}{4}l$.

Prove that $$f(x)=\frac{2l}{\pi^2}\left(\cos\frac{2\pi x}{l}+\frac{1}{9}\cos\frac{6\pi x}{l}+\frac{1}{25}\cos\frac{10\pi x}{l}\dots\right).$$

5. Show by expanding $\sin x$ in a cosine series that
$$\sin x=\frac{2}{\pi}\left(1-\frac{2\cos 2x}{1.3}-\frac{2\cos 4x}{3.5}-\frac{2\cos 6x}{5.7}\dots\right).$$

What function does the series represent when x lies between 0 and $-\pi$?

6. Show that, if u is a fraction,
$$\sin ux=\frac{2\sin u\pi}{\pi}\left\{\frac{\sin x}{1^2-u^2}-\frac{2\sin 2x}{2^2-u^2}+\frac{3\sin 3x}{3^2-u^2}-\frac{4\sin 4x}{4^2-u^2}\dots\right\}.$$

7. Prove that if $-\frac{\pi}{2}<x<\frac{\pi}{2}$,
$$\frac{\pi}{4}=\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\dots .$$

§ 66. It is known as the result of experiment, that if we have two parallel planes in a body, distant d apart, one of which is kept at a temperature v_1 and the other at a temperature v_2, v_1 being greater than v_2, the quantity of heat that flows across the slab between the planes, per unit area, per second, is

$$k\frac{v_1 - v_2}{d},$$

where k is a constant known as the conductivity of the substance for heat. Suppose that the axis of x is taken perpendicular to the two planes, and that they are brought close together so that d becomes dx and $v_1 - v_2$ becomes dv. Then $\frac{\partial v}{\partial x}$ gives the rate at which v increases with x at any point and $-k\frac{\partial v}{\partial x}$ the quantity of heat that flows per second through a unit of area drawn through the point with its normal in the direction of the x-axis. The minus sign is necessary because, if $\frac{\partial v}{\partial x}$ is positive, the flow takes place in the $-x$ direction.

The conductivity, k, is not strictly constant, but depends slightly on the temperature of the substance. In what follows, however, it will be considered constant.

§ 67. **Equation for the conduction of heat.**

Consider a rectangular element, the centre of which is situated at P(x, y, z) and the sides of which are dx, dy, dz. We shall find an expression for the rate at which heat is flowing into the element and shall equate it to the rate at which the quantity of heat in the element is increasing.

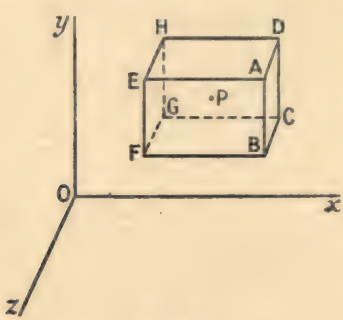

Fig. 46.

The "flow" of heat in the x direction at P is $-k\frac{\partial v}{\partial x}$. Its average value on face ABCD is

$$-k\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)\frac{dx}{2}.$$

Consequently the rate at which heat is flowing out of the element through ABCD is

$$-\left(k\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)\frac{dx}{2}\right)dy\,dz. \quad \text{.........................(7)}$$

The average value of the flow on face EFGH is

$$-k\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)\frac{dx}{2}$$

and the rate at which heat is flowing into the element through EFGH is

$$\left(-k\frac{\partial v}{\partial x}+\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)\frac{dx}{2}\right)dy\,dz \quad\text{.....................}(8)$$

On subtracting (7) from (8), we find that the rate at which heat is being gained by the element through the two "x faces," ABCD and EFGH, is

$$\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)dx\,dy\,dz.$$

Similarly, $\frac{\partial}{\partial y}\left(k\frac{\partial v}{\partial y}\right)dx\,dy\,dz$ and $\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\right)dx\,dy\,dz$ give respectively the rates at which heat is being gained through the y and z faces, and

$$\left\{\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\right)\right\}dx\,dy\,dz \quad\text{............}(9)$$

gives the total rate at which the element gains heat by conduction.

Let ρ be the density of the body and c its specific heat. Then the quantity of heat in the element is

$$c\rho v\,dx\,dy\,dz,$$

and the rate at which it is increasing is

$$c\rho\frac{\partial v}{\partial t}dx\,dy\,dz.$$

On equating this to (9) and cancelling out $dx\,dy\,dz$, we obtain

$$c\rho\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\right). \quad\text{.............}(10)$$

If we assume that the body is homogeneous, k does not vary with x, y and z, and may consequently be taken outside the differentiation. If κ be written for $k/(c\rho)$, the equation then assumes its usual form,

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \quad\text{or}\quad \frac{\partial v}{\partial t}=\kappa\nabla^2 v. \quad\text{..............}(11)$$

κ is called the diffusivity of the substance.

It is possible that heat may be created inside the element of volume, for example by the passage of an electric current through it, and in this case the equation requires modification. Let A be the quantity of heat created per second per unit of volume. Then the rate at which heat is being created inside the element is

$$A\,dx\,dy\,dz.$$

We must add this to (9). Hence the equation

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{A}{c\rho}. \quad\text{.....................}(12)$$

§ 68. **Equation for the conduction of heat. Otherwise.**

Take any closed surface inside the body, and let l, m, n be the direction cosines of the outward drawn normal to an element dS of this surface. The flow in the direction of the outward drawn normal is

$$-k\left(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}+n\frac{\partial v}{\partial z}\right).$$

The quantity of heat flowing in through dS per second is therefore

$$k\left(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}+n\frac{\partial v}{\partial z}\right)dS,$$

and consequently the rate at which heat is flowing into the region bounded by the surface is

$$\iint k\left(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}+n\frac{\partial v}{\partial z}\right)dS.$$

If we assume that k is constant and apply Gauss's theorem, this becomes

$$\iiint k\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)dx\,dy\,dz.$$

The rate at which the heat in the region bounded by the surface is increasing is given by

$$\iiint c\rho\frac{\partial v}{\partial t}\,dx\,dy\,dz.$$

The two volume integrals are equal, no matter what the shape of the surface is. Hence the integrands must be equal, whence equation.

If k is a function of x, y, z, Green's theorem must be used instead of Gauss's.

§ 69. **Equation for the conduction of heat in polars and cylindricals.**

It is sometimes necessary to express the equation for heat conduction in polar or cylindrical coordinates. The equation may be derived directly in these coordinates from first principles or it may be derived in generalised orthogonal coordinates and the proper substitutions made. Here we shall assume the result proved in § 24, that

$$\nabla^2 v=\frac{1}{\lambda\mu\nu}\Sigma\frac{\partial}{\partial\xi}\left(\frac{\mu\nu}{\lambda}\frac{\partial v}{\partial\xi}\right),$$

where ξ, η, ζ are the orthogonal coordinates and λ, μ, ν the multipliers.

On writing r, θ, ϕ for ξ, η, ζ and 1, r, $r\sin\theta$ for λ, μ, ν, we obtain for the equation of heat conduction in polar coordinates,

$$\frac{\partial v}{\partial t}=\frac{\kappa}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial v}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}\right)\right\}$$

$$=\frac{\kappa}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}\right\}, \quad\text{............(13)}$$

and on writing r, θ, z for ξ, η, ζ and 1, r, 1 for λ, μ, ν, we obtain the equation in cylindricals,

$$\frac{\partial v}{\partial t}=\frac{\kappa}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial v}{\partial z}\right)\right\}=\kappa\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}\right\}.$$

§ 70. **Boundary conditions.**

At the surface of separation of two media of different conductivities, k_1 and k_2, the temperature must be the same on both sides of the surface and as much heat must flow out of the one medium as flows into the other. Let v_1, v_2 denote the temperatures on different sides of the same element of the surface of separation and let n denote the direction of the normal to the element. Then, expressing these conditions mathematically, we obtain

$$v_1=v_2 \quad \text{and} \quad k_1\frac{\partial v_1}{\partial n}=k_2\frac{\partial v_2}{\partial n}.$$

At the surface of separation of a solid and a gas, it is usually assumed that the temperature of the gas is appreciably constant throughout, and that the Newtonian law of cooling holds, namely, there is a loss of heat from the surface of the solid proportional to the difference of temperature of the surface and the gas. If v denotes the temperature of the surface of the solid, v_0 the temperature of the gas, k the conductivity of the solid and n the direction of the normal to the surface drawn inwards, then

$$k\frac{\partial v}{\partial n}=e(v-v_0).$$

e is called the emissivity of the surface. It varies considerably with the condition and state of polish of the surface. It also varies with the temperature since the Newtonian law of cooling is strictly true only for small temperature differences.

If the surface is impervious to heat or is coated with a varnish impermeable by heat, $\frac{\partial v}{\partial n}=0.$

§ 71. **Uniqueness of solution of problem.**

When the initial and surface conditions are given, then the state of the body is fully determined for all subsequent times.

For, if possible, let there be two independent solutions v_1 and v_2. Then $\frac{\partial v}{\partial t}=\kappa\nabla^2 v$ throughout the solid,

$$v=f(x, y, z) \text{ for } t=0 \quad \text{and} \quad v=\phi(x, y, z, t) \text{ at the surface.}$$

Let $\mathsf{V}=v_1-v_2$. Then V satisfies $\frac{\partial \mathsf{V}}{\partial t}=\kappa\nabla^2\mathsf{V}$ throughout the solid,

$$\mathsf{V}=0 \text{ for } t=0 \quad \text{and} \quad \mathsf{V}=0 \text{ at the surface.}$$

We shall show that V must be zero everywhere in the solid. Consider the integral

$$J = \iiint \frac{V^2}{2}\, dx\, dy\, dz.$$

It gives $$\frac{\partial J}{\partial t} = \iiint V \frac{\partial V}{\partial t}\, dx\, dy\, dz = \kappa \iiint V \nabla^2 V\, dx\, dy\, dz$$

$$= \kappa \iiint V \frac{\partial V}{\partial n}\, dS - \kappa \iiint \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right\} dx\, dy\, dz,$$

by Green's theorem. The surface integral is taken over the surface of the body and the volume integrals throughout its volume.

Since $V = 0$ over the surface, $\iiint V \frac{\partial V}{\partial n}\, dS = 0$

and $$\frac{\partial J}{\partial t} = -\kappa \iiint \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right\} dx\, dy\, dz \leqq 0.$$

But $J = 0$ for $t = 0$.

Therefore $J \leqq 0$, *i.e.* $\iiint \frac{V^2}{2}\, dx\, dy\, dz \leqq 0.$

As V^2 cannot be negative, V must be zero everywhere. Hence $v_1 = v_2$, and we can have only one solution.

We shall now proceed to apply the differential equation for heat conduction to particular cases.

§ 72. Steady flow in one direction.

In the case of steady flow $\frac{\partial v}{\partial t} = 0$, and the equation becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Let the temperature be given for

$$x = 0 \quad \text{by} \quad v = V_0,$$

and for $x = d$ by $v = V_1$, for all values of t.

Then obviously the isothermals are planes parallel to the two given planes and v cannot vary with y or z. The equation becomes

$$\frac{\partial^2 v}{\partial x^2} = 0.$$

The integral of this is $v = Ax + B.$

On substituting the values for $x = 0$ and $x = d$, we obtain

$$V_0 = B, \quad V_1 = Ad + V_0.$$

Hence the solution is $$v = \frac{(V_1 - V_0)}{d} x + V_0.$$

This may be written $$\frac{v-V_0}{x}=\frac{V_1-V_0}{d};$$

hence the fall of temperature is proportional to the distance from $x=0$.

The quantity of heat that flows across area S of any isothermal in time t is given by

$$Q=-\iint k\frac{\partial v}{\partial x}t\,dS=\frac{k(V_0-V_1)St}{d}.$$

The quantity of heat contained in the slab bounded by the planes $x=0$ and $x=d$ is given by

$$\int_0^d c\rho Sv\,dx=c\rho S\left[\frac{(V_1-V_0)}{d}\frac{x^2}{2}+V_0x\right]_0^d=\frac{1}{2}c\rho Sd(V_0+V_1),$$

where S is the area of a face of the slab.

§ 73. **Steady flow. Symmetry about a point.**

We shall next suppose that the lines of flow radiate out from a point, which we shall take as origin, and that consequently the isothermals are concentric spheres with this point as centre. Let

$$v=V_a \text{ for } r=a \quad\text{and}\quad v=V_b \text{ for } r=b$$

for all values of t. Let b be greater than a.

In this case we take the equation of the conduction of heat in polar coordinates, and as v depends only on r, the equation reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)=0.$$

Integrating this, we obtain

$$r^2\frac{\partial v}{\partial r}=A,\quad \frac{\partial v}{\partial r}=\frac{A}{r^2},\quad v=-\frac{A}{r}+B.$$

Substitution of the values for $r=a$ and $r=b$ gives

$$V_a=-\frac{A}{a}+B,\quad V_b=-\frac{A}{b}+B.$$

Hence $$A=(V_b-V_a)\frac{ab}{b-a},\quad B=\frac{V_bb-V_aa}{b-a}$$

and $$v=(V_a-V_b)\frac{ab}{(b-a)}\frac{1}{r}+\frac{V_bb-V_aa}{b-a}.$$

The curve connecting r and v is therefore a rectangular hyperbola.

The quantity of heat that flows across any isothermal in time t is

$$-k\frac{\partial v}{\partial r}4\pi r^2t=4\pi kt(V_a-V_b)\frac{ab}{(b-a)}.$$

The quantity of heat contained between the isothermals $r=a$ and $r=b$ is

$$\int_a^b c\rho v 4\pi r^2\,dr = 4\pi\rho c\int_a^b\left\{(\mathrm{V}_a-\mathrm{V}_b)\frac{ab}{(b-a)}r+\frac{b\mathrm{V}_b-a\mathrm{V}_a}{b-a}r^2\right\}dr$$

$$=4\pi\rho c\left\{(\mathrm{V}_a-\mathrm{V}_b)\frac{ab(a+b)}{2}+(b\mathrm{V}_b-a\mathrm{V}_a)\frac{(a^2+ab+b^2)}{3}\right\}.$$

§ 74. **Two dimensions. Steady flow.**

(1) Suppose we have a thin plate (fig. 47) bounded by the lines $x=0$, $x=l$, $y=0$ and $y=\infty$, that the temperature on the edge $y=0$ is constant and given by $f(x)$, and that the temperature on the other edges is always zero. We shall also suppose that heat cannot escape from either surface of the plate and that the effect of initial conditions has passed away, that the temperature everywhere is independent of the time. The problem then becomes one of two-dimensional steady flow, and can be formulated as follows:

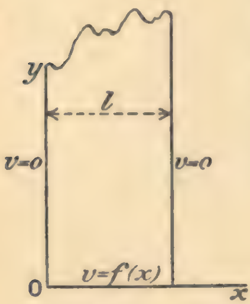

Fig. 47.

(1) $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0.$

(2) $v=0$ for $x=0$, $v=0$ for $x=l$,
$v=f(x)$ for $y=0$, $v=0$ for $y=\infty$.

Try $e^{\alpha x+\beta y}$ as a solution.

Then, since $\alpha^2+\beta^2=0$, either α or β must be imaginary. From the nature of the boundary conditions, since $v=0$ for $y=\infty$, β must be real and negative. Therefore our solution becomes

$$v=\mathrm{A}e^{-\beta y+i\beta x}+\mathrm{B}e^{-\beta y-i\beta x}\quad\text{or}\quad e^{-\beta y}(\mathrm{C}\cos\beta x+i\mathrm{D}\sin\beta x),$$

where A, B, C and D are constants. Since $v=0$ for $x=0$ or $x=l$, only the sine term can be used, and $\beta=\dfrac{m\pi}{l}$, where m is any integer. Giving m all its possible values and multiplying each term by a constant, we have therefore

$$v=\Sigma b_m e^{-\frac{m\pi y}{l}}\sin\frac{m\pi x}{l}.$$

We have now only to satisfy the condition $v=f(x)$ for $y=0$. We do this by fixing the values of b_m, by putting

$$b_m=\frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}\,dx.$$

Then, when $y=0$,

$$v=\Sigma b_m\sin\frac{m\pi x}{l},$$

i.e. the half range sine series for v. The solution is therefore

$$v = \Sigma b_m e^{-\frac{m\pi y}{l}} \sin \frac{m\pi x}{l},$$

where
$$b_m = \frac{2}{l}\int_0^l f(x) \sin \frac{m\pi x}{l}\, dx.$$

(2) Let us now take the analogous problem of steady flow in a rectangular plate bounded by $x=0$, $x=l$, $y=0$ and $y=h$, the boundary conditions being as follows:

$v=0$ for $x=0$, $\quad v=0$ for $x=l$,

$v=0$ for $y=0$, $\quad v=f(x)$ for $y=h$.

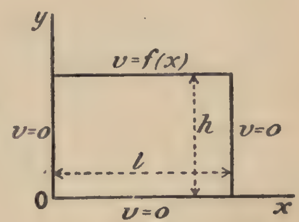

FIG. 48.

If we start with the same solution as before, $v=e^{\alpha x+\beta y}$, we find that, in order to satisfy the first and second conditions, it must take the form

$$v = e^{\pm\frac{m\pi y}{l}} \sin \frac{m\pi x}{l},$$

where m, as before, is any integer. In satisfying the third condition, we are apparently at a stop because neither $e^{+\frac{m\pi y}{l}}$ nor $e^{-\frac{m\pi y}{l}}$ vanishes when $y=0$. Their difference, however, vanishes. We have therefore

$$v = \sinh \frac{m\pi y}{l} \sin \frac{m\pi x}{l}.$$

Taking every possible value of m and multiplying each term by a constant b_m, we find for $y=h$,

$$v = \Sigma b_m \sinh \frac{m\pi h}{l} \sin \frac{m\pi x}{l}.$$

This must be the half range sine expansion for $f(x)$. Hence

$$b_m \sinh \frac{m\pi h}{l} = \frac{2}{l}\int_0^l f(x) \sin \frac{m\pi x}{l}\, dx.$$

If we include the $\sinh \frac{m\pi h}{l}$ in the b_m, the solution can be written

$$v = \Sigma b_m \frac{\sinh \frac{m\pi y}{l}}{\sinh \frac{m\pi h}{l}} \sin \frac{m\pi x}{l},$$

where
$$b_m = \frac{2}{l}\int_0^l f(x) \sin \frac{m\pi x}{l}\, dx.$$

(3) Let us now consider the case of the same rectangular plate with different boundary conditions, namely

$$v=0 \quad \text{for } x=0, \qquad v=0 \quad \text{for } x=l,$$
$$v=\phi(x) \text{ for } y=0, \qquad v=f(x) \text{ for } y=h.$$

To obtain a solution write $v=u+w$, where u satisfies the differential equation and the boundary conditions,

$$u=0 \text{ for } x=0, \qquad u=0 \quad \text{for } x=l,$$
$$u=0 \text{ for } y=0, \qquad u=f(x) \text{ for } y=h,$$

while w satisfies the differential equation and the boundary conditions,

$$w=0 \quad \text{for } x=0, \qquad w=0 \text{ for } x=l,$$
$$w=\phi(x) \text{ for } y=0, \qquad w=0 \text{ for } y=h.$$

Then $u+w$ satisfies the same boundary conditions as v. u satisfies the same conditions as v in (2), and is hence given by

$$u=\Sigma b_m \frac{\sinh \frac{m\pi y}{l}}{\sinh \frac{m\pi h}{l}} \sin \frac{m\pi x}{l},$$

where
$$b_m=\frac{2}{l}\int_0^l f(x) \sin \frac{m\pi x}{l}\, dx.$$

w satisfies the same conditions as v in (2), if the origin be shifted to the point 0, h and the direction of the y-axis be reversed. Hence

$$w=\Sigma b'_m \frac{\sinh \frac{m\pi (h-y)}{l}}{\sinh \frac{m\pi h}{l}} \sin \frac{m\pi x}{l},$$

where
$$b'_m=\frac{2}{l}\int_0^l \phi(x) \sin \frac{m\pi x}{l}\, dx.$$

EXAMPLES.

1. If the conductivity of copper be 0·97 for the calorie as unit of heat and the centimetre and second as units of length and time, find the value of the same conductivity when the lb., the degree Fahrenheit, the foot and the minute are taken as units.

2. Prove the following results for the steady flow of heat symmetrical about a straight line in an infinite solid.

Temperature at any point distant r from the axis of symmetry,

$$\frac{(V_a \log b - V_b \log a)-(V_a-V_b)\log r}{\log b - \log a},$$

where V_a and V_b are the temperatures of the cylindrical isothermals of radius a and b respectively.

Quantity of heat crossing any isothermal per unit length in time t,

$$\frac{2\pi kt(V_a - V_b)}{\log b - \log a}.$$

3. The inner and outer surfaces of a conducting shell are concentric spheres of radii r_1, r_2 and are maintained at constant temperatures v_1, v_2 respectively. If the conductivity of the substance is a linear function $f(v)$ of the temperature, show that the quantity of heat transmitted through the shell in unit time is the same as if the conductivity had the uniform value $f\{\frac{1}{2}(v_1 + v_2)\}$.

4. A hollow shell of isotropic material has conductivity $k_0 e^{-v/c}$, where k_0 and c are constants. The internal and external radii are a and b. Show that if the internal surface be maintained at temperature v_0 and the external surface at temperature zero, the heat conducted across the shell in unit of time is

$$4\pi k_0 \frac{ab}{b-a} c(1 - e^{-\frac{v_0}{c}}).$$

5. An infinitely long plane and uniform plate is bounded by two parallel edges and an end at right angles to these. The breadth is π, the end is maintained at temperature v_0 at all points and the edges at temperature zero. Show that the steady state as given by

$$v = \frac{4v_0}{\pi}\{e^{-y}\sin x + \tfrac{1}{3}e^{-3y}\sin 3x + \dots\},$$

where y is taken along one edge and x along the end from one corner as origin, satisfies all the conditions.

Identify this solution by any process with

$$v = 2\frac{v_0}{\pi}\tan^{-1}\frac{\sin x}{\sinh y}.$$

[Cf. Byerly's *Fourier's Series and Spherical Harmonics*, § 58.]

6. An infinitely long uniformly thick plate of homogeneous material is bounded by two parallel edges l apart, an end at right angles to the edges, and two plane faces which are coated with varnish impermeable by heat. The edges are maintained at temperature zero and the end is kept heated so that the temperature is V at the middle point and diminishes uniformly to zero at each edge.

Find approximately in terms of V the temperature on the middle line of the plate at a distance $3l/\pi$ from the heated end. Find also the rate of flow of heat across a cross-section at that point.

7. The two edges of an infinitely long rectangular plate are maintained at temperature zero while the end is maintained at temperature $v_0 \sin nx$. The breadth is π/n and x is measured along the end from one corner. Find an expression for the temperature when the steady state is established.

8. Find the temperature of the middle point of a thin square plate whose faces are impervious to heat when three edges are kept at the temperature 0° C. and the fourth edge at the temperature 100° C. (Answer 25° C.)

§ 75. **Variable linear flow. No radiation.**

(1) Let us suppose we have a bar of length l and of uniform section, the diameter of which is small in comparison with the radius of curvature. We shall also suppose that its surface is impervious to heat, that there is no radiation from the sides. Let the initial temperature of the bar be given and let its ends be kept at the constant temperature 0° C. Then, if one end of the bar be taken as origin and distances along the bar be denoted by x,

$$(1)\quad \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2};$$

$$(2)\quad \left.\begin{array}{l} v=0 \text{ when } x=0, \\ v=0 \text{ when } x=l, \end{array}\right\}\text{ for all values of } t;$$

$$(3)\quad v=f(x) \text{ for } t=0, \quad v\neq\infty \text{ for } t=\infty.$$

The boundary conditions suggest that x occurs in the solution as $\sin\frac{m\pi x}{l}$. Trying $e^{at}\sin\frac{m\pi x}{l}$, we find that this satisfies the differential equation if

$$a=-\kappa\left(\frac{m\pi}{l}\right)^2.$$

Taking every possible value of m and multiplying each term by a constant, b_m, we obtain

$$v=\Sigma b_m e^{-\kappa\left(\frac{m\pi}{l}\right)^2 t}\sin\frac{m\pi x}{l}.$$

This satisfies the condition for $t=\infty$ and for $t=0$ it reduces to the half range sine series. Hence the solution is

$$v=\Sigma b_m e^{-\kappa\left(\frac{m\pi}{l}\right)^2 t}\sin\frac{m\pi x}{l},$$

where
$$b_m=\frac{2}{l}\int^l f(x)\sin\frac{m\pi x}{l}\,dx.$$

We see from the solution that when $t=\infty$, $v=0$ everywhere; all the heat has escaped from the bar.

(2) Suppose, instead of the ends of the bar being kept at temperature zero, that they are impervious to heat. Then the statement of the problem becomes

$$(1)\quad \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2};$$

$$(2)\quad \left.\begin{array}{l} \dfrac{\partial v}{\partial x}=0 \text{ for } x=0, \\[2ex] \dfrac{\partial v}{\partial x}=0 \text{ for } x=l, \end{array}\right\}\text{ for all values of } t;$$

$$(3)\quad v=f(x) \text{ for } t=0, \quad v\neq\infty \text{ for } t=\infty.$$

The boundary condition suggests that $\cos\frac{m\pi x}{l}$ is a factor of the solution, and proceeding as before we obtain for the complete result

$$v = a_0 + \Sigma a_m e^{-\kappa\left(\frac{m\pi}{l}\right)^2 t}\cos\frac{m\pi x}{l},$$

where $$a_0 = \frac{1}{l}\int_0^l f(x)\,dx, \quad a_m = \frac{2}{l}\int_0^l f(x)\cos\frac{m\pi x}{l}\,dx.$$

We see from the solution that in this case when $t=\infty$, $v=a_0$, the average initial temperature. This result might have been inferred directly from the fact that no heat leaves the bar.

(3) Suppose that the ends are kept permanently at different temperatures, that

$$(1)\quad \frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2};$$

$$(2)\quad \left.\begin{aligned} v&=\beta \text{ for } x=0,\\ v&=\gamma \text{ for } x=l,\end{aligned}\right\}\text{ for all values of } t;$$

$$(3)\quad v=f(x) \text{ for } t=0, \quad v\neq\infty \text{ for } t=\infty.$$

Assume $v=u+w$, where $u=\phi(xt)$ and $w=\psi(x)$, and let w satisfy

$$(1)\quad \frac{\partial w}{\partial t} = \kappa\frac{\partial^2 w}{\partial x^2};$$

$$(2)\quad \left.\begin{aligned} w&=\beta \text{ for } x=0,\\ w&=\gamma \text{ for } x=l,\end{aligned}\right\}\text{ for all values of } t.$$

Since w is independent of t, from (1) it must have the form $w=\mathsf{A}x+\mathsf{B}$. From (2),

$$\beta=\mathsf{B}, \quad \gamma=\mathsf{A}l+\mathsf{B}.$$

Hence $$w=\frac{(\gamma-\beta)}{l}x+\beta.$$

The conditions, which u must satisfy, are then

$$(1)\quad \frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2};$$

$$(2)\quad \left.\begin{aligned} u&=0 \text{ for } x=0,\\ u&=0 \text{ for } x=l,\end{aligned}\right\}\text{ for all values of } t;$$

$$(3)\quad u=f(x)-\frac{(\gamma-\beta)}{l}x-\beta \text{ for } t=0, \quad u\neq\infty \text{ for } t=\infty.$$

But this is the same as the first problem in this section. Hence the complete solution for v is

$$v=\frac{(\gamma-\beta)}{l}x+\beta+\Sigma b_m e^{-\kappa\left(\frac{m\pi}{l}\right)^2 t}\sin\frac{m\pi x}{l},$$

where $$b_m=\frac{2}{l}\int_0^l\left\{f(x)-\frac{(\gamma-\beta)}{l}x-\beta\right\}\sin\frac{m\pi x}{l}\,dx.$$

§ 76. **Equation for variable linear flow with radiation.**

Consider a thin rod of uniform cross-sectional area σ situated in air at temperature zero. Let k be the conductivity of the rod, ρ its density, c its specific heat, p its perimeter and e the emissivity of its surface. Let x denote distance measured along the axis of the rod and let the isothermal surfaces be planes perpendicular to the axis of the rod.

Consider an element of the rod bounded by the planes x and $x+dx$. The rate at which heat is being conducted into it is

$$-\sigma k\frac{\partial v}{\partial x}.$$

The rate at which heat is being conducted out of it is

$$-\sigma k\left(\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}dx\right).$$

Hence the rate of gain by conduction is

$$\sigma k\frac{\partial^2 v}{\partial x^2}dx.$$

The temperature of the element is v, the area of its surface is $p\,dx$, and hence the rate at which it loses heat by radiation is

$$evp\,dx.$$

The quantity of heat in the element is $vc\rho\sigma\,dx$ and the rate at which it is increasing is

$$\frac{\partial v}{\partial t}c\rho\sigma\,dx.$$

We arrive therefore at the equation

$$\frac{\partial v}{\partial t}c\rho\sigma\,dx=\sigma k\frac{\partial^2 v}{\partial x^2}dx-evp\,dx \quad \text{or} \quad \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-hv,$$

where

$$h=\frac{ep}{c\rho\sigma}.$$

In order that the isothermals may be planes, it is necessary that the rate at which heat is being conducted out of the element should be much greater than the rate at which it is being radiated out.

§ 77. **Ingenhousz's experiment.**

Suppose that the flow is steady, that one end of the bar is situated at the origin and has a fixed temperature, while the other end is at infinity and has temperature zero. Then

$$(1)\quad \frac{\partial^2 v}{\partial x^2}-\frac{ep}{k\sigma}v=0;$$

$$(2)\quad v=\mathsf{V}\text{ for }x=0,\quad v=0\text{ for }x=\infty.$$

The solution of (1) is $v=\mathsf{A}e^{\sqrt{\frac{ep}{k\sigma}}x}+\mathsf{B}e^{-\sqrt{\frac{ep}{k\sigma}}x}$.

From (2), we find that $A=0$, $B=V$. Hence

$$v=Ve^{-\sqrt{\frac{ep}{k\sigma}}x}.$$

A well known experiment is to coat two similar bars of different metals with wax, and to fix them up parallel with one end of each projecting into a vessel in which water can be boiled. The bars are long enough for the cold ends to be near the temperature of the atmosphere, when the steady state is reached. Let the conductivities of the two bars be k_1, k_2 and let the distances along which the wax is melted be l_1, l_2. Then, where the wax just melts, the temperature must be the same on each bar. Therefore

$$\sqrt{\frac{ep}{k_1\sigma}}\,l_1=\sqrt{\frac{ep}{k_2\sigma}}\,l_2 \quad \text{or} \quad \frac{k_1}{k_2}=\frac{l_1^2}{l_2^2},$$

since e, p and σ are the same for both bars.

§ 78. **Despretz' formula.**

One end of a bar is kept at a constant temperature V and heat is conducted along the bar and escapes by radiation into the air. Then, when the steady state is established,

$$v=Ve^{-\sqrt{\frac{ep}{k\sigma}}x},$$

as in last section. Let v_1, v_2, v_3 be respectively the temperatures at $x-d$, x and $x+d$. Then

$$v_1=Ve^{-\sqrt{\frac{ep}{k\sigma}}(x-d)}, \quad v_2=Ve^{-\sqrt{\frac{ep}{k\sigma}}x} \quad \text{and} \quad v_3=Ve^{-\sqrt{\frac{ep}{k\sigma}}(x+d)}.$$

Denote $\frac{v_1+v_3}{v_2}$ by $2n$. Then

$$2n=e^{\sqrt{\frac{ep}{k\sigma}}d}+e^{-\sqrt{\frac{ep}{k\sigma}}d},$$

whence

$$e^{\sqrt{\frac{ep}{k\sigma}}d}=n+\sqrt{n^2-1},$$

the root with the minus sign being impossible. Therefore

$$\sqrt{\frac{ep}{k\sigma}}\,d=\log_e(n+\sqrt{n^2-1}).$$

We can determine n experimentally. Hence, in the comparison of two bars of different materials, if e, p, σ and d be the same for both and k_1, n_1 refer to the one bar and k_2, n_2 to the other,

$$\sqrt{\frac{k_1}{k_2}}=\frac{\log_e(n_2+\sqrt{n_2^2-1})}{\log_e(n_1+\sqrt{n_1^2-1})}.$$

§ 79. Consider the case of a finite rod of length l, from which there is radiation, both ends of which are kept at zero temperature and the temperature of which is initially given. The statement of the problem is as follows:

$$(1)\quad \frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} - hv;$$

$$(2)\quad v = 0 \quad \text{for } x = 0, \quad v = 0 \text{ for } x = l;$$

$$(3)\quad v = f(x) \text{ for } t = 0, \quad v \neq \infty \text{ for } t = \infty.$$

Write $v = e^{-ht}u$. Then, by substituting in (1), we find that it reduces to

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

The boundary and initial conditions are the same for u and v. By comparison with § 75, it will be seen that the problem has been reduced to the analogous one with no radiation. The solution is therefore

$$v = \Sigma b_m e^{-\left\{h+\kappa\left(\frac{m\pi}{l}\right)^2\right\}t} \sin \frac{m\pi x}{l},$$

where

$$b_m = \frac{2}{l}\int_0^l f(x) \sin \frac{m\pi x}{l}\, dx.$$

§ 80. **Fourier's ring.**

Suppose we have a thin bar of uniform section bent into the form of a circular ring of radius a. At one point, O, in the ring let a steady temperature be maintained and let heat be radiated from the ring to the air. It is required to find the temperature of the ring when the steady state is established.

Take O as the origin and denote the distance from O, measured round the ring, by x. Then

$$(1)\quad \frac{\partial^2 v}{\partial x^2} - \frac{ep}{k\sigma} v = 0;$$

$$(2)\quad v = \mathsf{V} \text{ for } x = 0, \quad \frac{\partial v}{\partial x} = 0 \text{ for } x = \pm \pi a.$$

The solution of (1) is

$$v = \mathsf{A}e^{\sqrt{\frac{ep}{k\sigma}}x} + \mathsf{B}e^{-\sqrt{\frac{ep}{k\sigma}}x}.$$

From (2), we have

$$\mathsf{V} = \mathsf{A} + \mathsf{B}, \quad 0 = \mathsf{A}e^{\sqrt{\frac{ep}{k\sigma}}\pi a} - \mathsf{B}e^{-\sqrt{\frac{ep}{k\sigma}}\pi a},$$

whence

$$\mathsf{A} = \frac{\mathsf{V}e^{-\sqrt{\frac{ep}{k\sigma}}\pi a}}{2\cosh\sqrt{\frac{ep}{k\sigma}}\pi a}, \quad \mathsf{B} = \frac{\mathsf{V}e^{+\sqrt{\frac{ep}{k\sigma}}\pi a}}{2\cosh\sqrt{\frac{ep}{k\sigma}}\pi a}.$$

The solution is therefore

$$v=\frac{\mathsf{V}\left\{e^{\sqrt{\frac{ep}{k\sigma}}(x-\pi a)}+e^{-\sqrt{\frac{ep}{k\sigma}}(x-\pi a)}\right\}}{2\cosh\sqrt{\frac{ep}{k\sigma}}\pi a}=\frac{\mathsf{V}\cosh\sqrt{\frac{ep}{k\sigma}}(x-\pi a)}{\cosh\sqrt{\frac{ep}{k\sigma}}\pi a}.$$

The heat radiated from the ring into the air in time t is

$$2t\int_0^{\pi a} epv\,dx=2\mathsf{V}t\sqrt{epk\sigma}\tanh\sqrt{\frac{ep}{k\sigma}}\pi a.$$

§ 81. **Linear flow in semi-infinite solid. Temperature on face given as harmonic function of the time.**

Let all space on the positive side of the yz-plane be filled with a homogeneous solid of diffusivity κ. Let the temperature on the yz-plane be given as a harmonic function of the time and let it be the same for all values of y and z. It is required to find the temperature throughout the solid when the periodic state is established.

Clearly v is here independent of y, z, and the conditions to be fulfilled are as follows:

$$(1)\ \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2};$$

$$(2)\ v=\mathsf{V}\sin nt \text{ for } x=0,\quad v\neq\infty \text{ for } x=\infty.$$

Try $e^{\alpha t+\beta x}$. Then, in order to satisfy (1), $\alpha=\kappa\beta^2$. Before, in § 75, we used the special case of this solution when α was real and negative. The form then obtained is, however, not suited to the present case. Suppose here that α is imaginary, *i.e.* try $\alpha=\pm i\gamma$. Then we obtain

$$e^{\pm i\gamma t\pm\sqrt{\pm\frac{i\gamma}{\kappa}}x}.$$

Now $\qquad (1+i)^2=2i \quad\text{and}\quad (1-i)^2=-2i.$

Therefore $\qquad \sqrt{i}=\frac{1}{\sqrt{2}}(1+i)\quad\text{and}\quad\sqrt{-i}=\frac{1}{\sqrt{2}}(1-i).$

Hence the expression becomes

$$e^{\pm i\gamma t\pm\sqrt{\frac{\gamma}{2\kappa}}(1\pm i)x}=e^{\pm\sqrt{\frac{\gamma}{2\kappa}}x\pm i\left(\gamma t\pm\sqrt{\frac{\gamma}{2\kappa}}x\right)}.$$

This satisfies the differential equation. In order to satisfy the condition for $x=\infty$, the sign before the root must be $-$. We may now write the expression

$$e^{-\sqrt{\frac{\gamma}{2\kappa}}x}\left[\mathsf{A}e^{+i\left(\gamma t-\sqrt{\frac{\gamma}{2\kappa}}x\right)}+\mathsf{B}e^{-i\left(\gamma t-\sqrt{\frac{\gamma}{2\kappa}}x\right)}\right],$$

where A and B are constants. The condition for $x=0$ requires that the square bracket should take the form $\sin\left(\gamma t-\sqrt{\frac{\gamma}{2\kappa}}x\right)$, with $\gamma=n$.

The final solution is thus

$$v = \mathsf{V}e^{-\sqrt{\frac{n}{2\kappa}}x}\sin\left(nt - \sqrt{\frac{n}{2\kappa}}x\right). \qquad (14)$$

This result has an important application to the determination of the conductivity of the earth's crust.

The diurnal variation of the temperature of the earth's surface cannot be traced below a depth of 3-4 feet, the annual variation cannot be traced beyond a depth of 60-70 feet. As far as they are concerned, the convexity of the earth's surface may be neglected, and we may regard the phenomenon as the propagation of a plane wave into an earth with a plane surface.

Let x denote distance from the surface measured positive downwards and let the maximum diurnal or annual variations be measured for two depths, x_1 and x_2. Let the results obtained be v_1 and v_2. Then

$$\frac{v_1}{v_2} = e^{-\sqrt{\frac{n}{2\kappa}}(x_1 - x_2)}.$$

In this expression $n = 2\pi/\mathsf{T}$, where T is either 1 day or 365 days, according to the case chosen. Everything is known except κ, and hence κ can be determined.

We can determine from (14) the ratio of the depths at which the annual and diurnal variations are just perceptible. For, denote these depths by x_1 and x_2, and let the values of the mean annual and daily surface variation of temperature be A and D. Then

$$\mathsf{A}e^{-\sqrt{\frac{\pi}{365\kappa}}x_1} = \mathsf{D}e^{-\sqrt{\frac{\pi}{\kappa}}x_2},$$

and x_1/x_2 can be calculated when A/D is known.

Of course the above theory is an approximate one. Neither the annual nor the diurnal variation can be represented as a simple sine curve. But they can be represented by Fourier series, of which these sines are the most important terms, and the approximation improves as we descend into the earth owing to the higher terms of the series dying away more rapidly.

§ 82. Ångström's method of determining the conductivity of bars.

In this method, which according to Lord Kelvin is the best yet devised, the middle of the bar is subjected to a periodic heating and cooling, and measurements are made on the velocity of the heat waves along the bar and the rate of decay of their amplitude. The conditions may be stated approximately as follows:

(1) $\dfrac{\partial v}{\partial t} = \kappa\dfrac{\partial^2 v}{\partial x^2} - hv$;

(2) $v = \mathsf{V}\sin nt$ for $x = 0$, $\quad v \neq \infty$ for $x = \infty$,

the sole difference from the preceding section being the radiation term in (1).

Assume as a solution $v = \mathsf{V}e^{-gx}\sin(nt - fx)$.

We find on substituting and equating the coefficients of the sine and cosine terms to zero, that

$$\kappa(g^2 - f^2) - h = 0 \quad \text{and} \quad n - 2\kappa gf = 0.$$

Now g and f are determined from the observations and n is known. Hence κ is given by

$$\kappa = \frac{n}{2gf}.$$

EXAMPLES.

1. A rod is surrounded by a medium at temperature zero and its two ends are maintained at a constant temperature V. Show that, when a steady state has been reached, the temperature at the middle point will be $\mathsf{V}\operatorname{sech}\left(l\sqrt{\frac{ep}{k\sigma}}\right)$, where $2l$ is the length of the rod, k its conductivity, p the perimeter, σ the area of the cross-section and e the emissivity of the surface.

2. Show that the series $\frac{8\gamma}{\pi^2}\sum_1^\infty \frac{1}{m^2}\sin\frac{m\pi x}{l}\sin\frac{m\pi}{2}$ has the value $2x\gamma/l$ for $0 < x < l/2$ and $2(l-x)\gamma/l$ for $l/2 < x < l$.

Apply the result to the problem of the temperature distribution at time t in a bar of length l, the ends of which are kept at zero temperature, and in which the temperature originally increased uniformly from zero at one end to the middle point and thence diminished uniformly to zero at the other end. The lateral surface of the bar is impervious to heat.

3. It has been proposed to represent the rate of cooling of a surface by the empirical formula $e(v - v_0)^n$, where n has the value 1·2. Show that on this supposition the equation to be satisfied in a long thin rod cooling laterally is

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} - \frac{ep}{c\rho\sigma}(v - v_0)^n,$$

where v is the temperature at distance x measured along the rod from one end, v_0 is the temperature of the medium, κ is the diffusivity, ρ the density, p the perimeter, c the specific heat and σ the area of cross-section of the rod.

4. A bar of length l is heated so that its two ends are at temperature zero. If initially the temperature is given by

$$v = \frac{cx(l-x)}{l^2},$$

show that the temperature at time t at any point is given by

$$v = \frac{8ce^{-ht}}{\pi^3}\left\{e^{-\frac{\pi^2\kappa t}{l^2}}\sin\frac{\pi x}{l} + \frac{1}{3^2}e^{-\frac{3^2\pi^2\kappa t}{l^2}}\sin\frac{3\pi x}{l} + \dots\right\}.$$

5. A uniform cylindrical bar, of length l and small cross-section, is kept at a constant temperature v_0 at one end and placed in a medium at temperature zero. If the temperature at a distance x from the end is v_0e^{-ax} in the steady state, prove that the product of half the radius of the bar into the ratio of the conductivity to the emissivity is a^{-2}.

6. Two iron slabs each 20 cm. thick, one of which is at the temperature 0° and the other at the temperature 100° throughout, are placed together face to face, and their outer faces are kept at the temperature 0°. Find the temperature of a point in their common face and of points 10 cm. from the common face fifteen minutes after the slabs have been put together. Given $\kappa = 0{\cdot}24$ in C.G.S. units.

7. Show that the equation for the conduction of heat in a thin wire in which an electric current of constant strength γ is flowing, is given by

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} - hv + \frac{\gamma^2}{c\rho\sigma^2 \mathrm{C}},$$

where γ is the current and C the electrical conductivity, *i.e.* the reciprocal of the resistance per unit cross-section per unit length.

The surface of a uniform wire is impervious to heat, the ends are at the same temperature and the current γ has been flowing long enough for the steady state to be established. Show that the ratio of the thermal and electrical conductivities is given by

$$\frac{k}{\mathrm{C}}(v_2 - v_1) = \frac{1}{2}(u_1 - u_2)^2,$$

where v_2 and u_2 are respectively the temperature and potential at the middle of the wire and v_1, u_1 their values for one end.

8. At depths of 6, 12, 24 feet the annual ranges of fluctuation of temperature are 5·6° C., 2·8° C., 0·7° C. Find the velocity of propagation of the temperature wave into the earth.

9. A solid is bounded by the planes $x=0$ and $x=l$. Discuss the following cases, where the surface temperatures have been kept at the given values so long that the distribution of temperature in the solid is purely periodic:

(i) $x=0$ at $v=a+b\sin pt$: $x=l$ at zero.

(ii) $x=0$ at $v=a+b\sin pt$: $x=l$, impervious to heat.

(iii) $x=0$ and $x=l$ at $v=a+b\sin pt$.

(iv) $x=0$ at $v=a+b\sin pt$: $x=l$ at $v=a-b\sin pt$.

10. A large ring, of uniform cross-section small in every dimension, is heated initially so that there is a uniform gradient of temperature round each half from one point to the diametrically opposite point; it is then left to itself in a medium at zero temperature. Find the distribution of temperature in the ring at any subsequent time.

11. A thin ring surrounded by a medium at temperature zero is heated at one point by a source of temperature V_0. After the temperature of the ring has assumed a steady condition, the source is withdrawn. Express by means of a Fourier series the value of the temperature at any point of the ring at a time t after the suppression of the source.

12. A thin uniform ring of radius a has initially one half of its length at temperature v_0 and the other half at temperature zero, and is left to itself in air at temperature zero: find a trigonometric series to express the distribution of temperature.

13. Show that after time t the mean temperature of the ring in the preceding question is $\frac{v_0}{2}e^{-\frac{ep}{c\rho\sigma}t}$, in which c is the specific heat, ρ the density of the material, σ the cross-sectional area, p the perimeter and e the emissivity of the surface.

§ 83. **Flow of heat in a sphere. Surface at zero temperature.**

Let the radius of the sphere be a and let its initial temperature be given by $v=f(r)$. Then, from considerations of symmetry, flow must take place only in the direction of the radius. The equation (cf. § 69) becomes therefore

$$\frac{\partial v}{\partial t}=\frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right).$$

Write $rv=u$. Then

$$\frac{\partial v}{\partial r}=\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2},\quad r^2\frac{\partial v}{\partial r}=r\frac{\partial u}{\partial r}-u\quad\text{and}\quad\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)=r\frac{\partial^2 u}{\partial r^2}.$$

Hence the conditions that u must satisfy are

(1) $\dfrac{\partial u}{\partial t}=\kappa\dfrac{\partial^2 u}{\partial r^2}$;

(2) $u=0$ for $r=0$, $u=0$ for $r=a$;

(3) $u=rf(r)$ for $t=0$, $u\neq\infty$ for $t=\infty$.

The problem is thus mathematically the same as that of § 75, (1).

§ 84. **Linear flow in doubly-infinite solid. Fourier's integral.**

(1) $\dfrac{\partial v}{\partial t}=\kappa\dfrac{\partial^2 v}{\partial x^2}$;

(2) $v\neq\infty$ for $x=\pm\infty$ for all values of t;

(3) $v=f(x)$, $-\infty<x<+\infty$, for $t=0$; $v\neq\infty$ for $t=\infty$.

The new feature in this problem is that the temperature is initially prescribed over an infinite instead of a finite range.

Suppose that instead of extending to infinity both ways, the solid extends only to $\pm l$. Then, as in § 75, the particular solution is

$$e^{-\kappa\left(\frac{m\pi}{l}\right)^2 t}\begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{m\pi x}{l}\right).$$

If we take every possible value of m and multiply as usual the cosine terms by a_m and the sine terms by b_m, then when $t=0$, the solution will take the form

$$v=a_0+\Sigma a_m\cos\frac{m\pi x}{l}+\Sigma b_m\sin\frac{m\pi x}{l}.$$

This satisfies the initial conditions if

$$a_0=\frac{1}{2l}\int_{-l}^{+l}f(x)\,dx,$$

$$a_m=\frac{1}{l}\int_{-l}^{+l}f(x)\cos\frac{m\pi x}{l}\,dx\quad\text{and}\quad b_m=\frac{1}{l}\int_{-l}^{+l}f(x)\sin\frac{m\pi x}{l}\,dx.$$

Substitute ξ for x in the formulae for a_0, a_m and b_m, and write the values of a_0, a_m and b_m in the series. Then

$$f(x)=\frac{1}{2l}\int_{-l}^{+l}f(\xi)\,d\xi+\Sigma\frac{1}{l}\int_{-l}^{+l}f(\xi)\cos\frac{m\pi\xi}{l}\,d\xi\cos\frac{m\pi x}{l}$$

$$+\Sigma\frac{1}{l}\int_{-l}^{+l}f(\xi)\sin\frac{m\pi\xi}{l}\,d\xi\sin\frac{m\pi x}{l}$$

$$=\frac{1}{l}\int_{-l}^{+l}f(\xi)\left[\frac{1}{2}+\Sigma\cos\frac{m\pi\xi}{l}\cos\frac{m\pi x}{l}+\Sigma\sin\frac{m\pi\xi}{l}\sin\frac{m\pi x}{l}\right]d\xi$$

$$=\frac{1}{l}\int_{-l}^{+l}f(\xi)\left[\frac{1}{2}+\Sigma\cos\frac{m\pi}{l}(\xi-x)\right]d\xi$$

$$=\frac{1}{2l}\int_{-l}^{+l}f(\xi)\left[1+\Sigma\cos\left\{\frac{m\pi}{l}(\xi-x)\right\}+\Sigma\cos\left\{-\frac{m\pi}{l}(\xi-x)\right\}\right]d\xi$$

$$=\frac{1}{2\pi}\int_{-l}^{+l}f(\xi)\left[\frac{\pi}{l}+\Sigma\frac{\pi}{l}\cos\left\{\frac{m\pi}{l}(\xi-x)\right\}+\Sigma\frac{\pi}{l}\cos\left\{-\frac{m\pi}{l}(\xi-x)\right\}\right]d\xi.$$

When l is made infinitely large, the square bracket becomes

$$\int_{-\infty}^{+\infty}\cos a\,(\xi-x)\,da,$$

assuming that the integral is convergent, and thus

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}f(\xi)\cos a\,(\xi-x)\,da.$$

This is Fourier's integral. It is the form which the series takes when the range is made infinite both ways, and it is equal to $f(x)$ throughout the range. As we have derived it from Fourier's series, $f(x)$ must here be subject to the same conditions as are necessary for its expansion in a series.

If we return now to our problem, we see that its solution is

$$v=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}f(\xi)e^{-\kappa a^2t}\cos a\,(\xi-x)\,da.$$

This result may be put into another form Changing the sign of a does not alter the value of $e^{-\kappa a^2t}\cos a\,(\xi-x)$, also there is a well known definite integral,

$$\int_0^{\infty}e^{-az^2}\cos bz\,dz=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-\frac{b^2}{4a}};$$

hence

$$\int_{-\infty}^{+\infty}e^{-\kappa a^2t}\cos a\,(\xi-x)\,da=2\int_0^{\infty}e^{-\kappa a^2t}\cos a\,(\xi-x)\,da=\sqrt{\frac{\pi}{\kappa t}}\,e^{-\frac{(\xi-x)^2}{4\kappa t}}$$

and
$$v=\frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{+\infty}f(\xi)\,e^{-\frac{(\xi-x)^2}{4\kappa t}}\,d\xi.$$

§ 85. **Other forms of Fourier's integral.**

We have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}f(\xi)\cos\alpha(\xi-x)\,d\alpha=\frac{1}{\pi}\int_{-\infty}^{+\infty}d\xi\int_{0}^{\infty}f(\xi)\cos\alpha(\xi-x)\,d\alpha,$$

since altering the sign of α does not change the sign of $\cos\alpha(\xi-x)$.

Suppose that it is desired to represent $f(x)$ only from $x=0$ to $x=+\infty$. Then we can give it any arbitrary form from $x=-\infty$ to $x=0$. The integral can be written in the form

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}d\xi\int_{0}^{\infty}f(\xi)(\cos\alpha\xi\cos\alpha x+\sin\alpha\xi\sin\alpha x)\,d\alpha.$$

Complete $f(x)$ in the range $x=-\infty$ to $x=0$ so that $f(-x)=f(x)$, so that it is an even function of x. Then, if we integrate first with respect to ξ the $\sin\alpha\xi$ term will vanish, since its sign changes with the sign of ξ, and the $\cos\alpha\xi$ term will give the same result from $-\infty$ to 0 as from 0 to $+\infty$. Thus the integral becomes

$$\frac{2}{\pi}\int_{0}^{\infty}d\xi\int_{0}^{\infty}f(\xi)\cos\alpha\xi\cos\alpha x\,d\alpha.$$

On the other hand, if we assume that $f(x)$ is filled in on the negative half of the range so that $f(-x)=-f(x)$, the integral becomes

$$\frac{2}{\pi}\int_{0}^{\infty}d\xi\int_{0}^{\infty}f(\xi)\sin\alpha\xi\sin\alpha x\,d\alpha.$$

These two integrals are analogous respectively to the half range cosine and sine series; and they might have been derived directly from the latter.

§ 86. **Linear flow in semi-infinite solid.**

We shall assume that the temperature on the face of the solid is zero. Then the statement of the problem is

$$(1)\quad \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2};$$

$$(2)\quad v=0 \text{ for } x=0, \qquad v\neq\infty \text{ for } x=\infty;$$

$$(3)\quad v=f(x) \text{ for } t=0, \quad v\neq\infty \text{ for } t=\infty.$$

As in § 84, the particular solution is of the type

$$e^{-\kappa\alpha^2 t}\,{\sin \atop \cos}\,\alpha x.$$

The sine must be chosen in order to satisfy the condition for $x=0$. Hence the solution is

$$v=\frac{2}{\pi}\int_{0}^{\infty}d\xi\int_{0}^{\infty}f(\xi)e^{-\kappa\alpha^2 t}\sin\alpha\xi\sin\alpha x\,d\alpha.$$

This expression may be put in the form

$$v=\frac{1}{\pi}\int_0^\infty d\xi\int_0^\infty f(\xi)e^{-\kappa\alpha^2 t}[\cos\alpha(\xi-x)-\cos\alpha(\xi+x)]\,d\alpha.$$

With the aid of the formula

$$\int_0^\infty e^{-az^2}\cos bz\,dz=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-\frac{b^2}{4a}},$$

this reduces to

$$v=\frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty f(\xi)\left[e^{-\frac{(\xi-x)^2}{4\kappa t}}-e^{-\frac{(\xi+x)^2}{4\kappa t}}\right]d\xi.$$

If the face of the solid is impervious to heat instead of being at the temperature zero, we must take the cosine integral.

§ 87. **The age of the earth.**

In the preceding problem let $f(x)$ be constant and $=c$. Then

$$v=\frac{c}{2\sqrt{\pi\kappa t}}\left[\int_0^\infty e^{-\frac{(\xi-x)^2}{4\kappa t}}d\xi-\int_0^\infty e^{-\frac{(\xi+x)^2}{4\kappa t}}d\xi\right]$$

$$=\frac{c}{2\sqrt{\pi\kappa t}}\left[\int_{-x}^\infty e^{-\frac{\beta^2}{4\kappa t}}d\beta-\int_{+x}^\infty e^{-\frac{\beta^2}{4\kappa t}}d\beta\right]=\frac{c}{\sqrt{\pi\kappa t}}\int_0^x e^{-\frac{\beta^2}{4\kappa t}}d\beta,$$

since $e^{-\frac{\beta^2}{4\kappa t}}$ is an even function of β. Write $\alpha^2=\beta^2/(4\kappa t)$; then the result takes the simpler form

$$v=\frac{2c}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\kappa t}}}e^{-\alpha^2}d\alpha. \qquad (15)$$

Tables of values of this integral have been drawn up, and hence v can be determined as a function of the upper limit.

If we descend into the earth we find that after we pass the points where the diurnal and annual variation cease to be appreciable, the temperature begins to increase. The rate of increase varies from place to place, but may be taken roughly as 1° F. for every 50 feet of descent for depths up to about 1 mile. This increase of temperature is easily explained on the assumption that the centre of the earth is at a high temperature and that heat is flowing outwards.

If we assume that the earth was originally at a uniform temperature c and that its surface has been always at a constant temperature zero, we can use the above result to find how long it has taken to cool. We neglect the convexity of the earth's surface.

We find from (15) that

$$\frac{\partial v}{\partial x}=\frac{2c}{\sqrt{\pi}}e^{-\frac{x^2}{4\kappa t}}\frac{1}{2\sqrt{\kappa t}}=\frac{c}{\sqrt{\pi\kappa t}}e^{-\frac{x^2}{4\kappa t}}.$$

Kelvin found by the method indicated in § 81 that κ for the material of the earth's surface has the value 400, the units of length and time being the foot and year. Assume that the earth was initially at the temperature of molten rock, *i.e.* about 7000° F. Insert the value for the gradient at the surface, namely, $\frac{dv}{dx} = 1°$ F. for every 50 feet. Then, writing $x = 0$ in the exponential, we obtain

$$\frac{\sqrt{t}}{50} = \frac{7000}{\sqrt{\pi 400}},$$

i.e.

$$t = \frac{7000^2 \times 50^2}{400\pi} = 10^8 \text{ years.}$$

If we write $x = 100$ miles,

$$e^{-\frac{x^2}{4\kappa t}} = e^{-\frac{(100 \times 5280)^2}{1600 \times 10^8}} = e^{-\frac{3}{2}}.$$

At that depth the gradient is only $\frac{1}{4 \cdot 5}$ of its surface value after 10^8 years. We see, therefore, from the first result, that according to our assumptions 10^8 years have elapsed since the earth was at a temperature of 7000° F., and we see from the second result that it is permissible to neglect the convexity of the earth's surface. The assumption throughout all the temperature change of constant conductivity, specific heat and density is, of course, open to question. Also the earth may have taken much longer to cool, owing to the liberation of heat due to the radio-active disintegration of some of its material.

§ 88. **Point source of heat.**

Consider the expression

$$v = \frac{Q}{8c\rho(\pi\kappa t)^{\frac{3}{2}}} e^{-\frac{r^2}{4\kappa t}}. \quad \text{..........................(16)}$$

It may be shown by trial to satisfy the equation for heat conduction when there is symmetry about the origin, namely,

$$\frac{\partial v}{\partial t} = \frac{\kappa}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right).$$

It therefore represents heat flowing to or from the origin.

The total quantity of heat in the field is given by

$$\int_0^\infty c\rho v 4\pi r^2\, dr = \frac{\pi Q}{2(\pi\kappa t)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{r^2}{4\kappa t}} r^2 dr.$$

Now we have the well-known result

$$\int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}.$$

On integrating by parts,

$$\int_0^\infty e^{-z^2}dz = \left(e^{-z^2}z\right)_0^\infty + 2\int_0^\infty e^{-z^2}z^2\,dz.$$

Therefore
$$\int_0^\infty e^{-z^2}z^2\,dz = \frac{\sqrt{\pi}}{4}, \text{ since } \underset{z=\infty}{\mathrm{Lt}} \frac{z}{e^{z^2}} = 0.$$

Hence the total quantity of heat in the field

$$\frac{4Q}{\sqrt{\pi}}\int_0^\infty e^{-\frac{r^2}{4\kappa t}}\frac{r^2}{4\kappa t}\frac{dr}{\sqrt{4\kappa t}} = Q,$$

and is constant.

The temperature

$$v = \frac{Q}{8c\rho(\pi\kappa)^{\frac{3}{2}}}t^{-\frac{3}{2}}e^{-\frac{r^2}{4\kappa t}} = \frac{Q}{8c\rho(\pi\kappa)^{\frac{3}{2}}}e^{-\frac{3}{2}\log t - \frac{r^2}{4\kappa t}}.$$

If r is not 0 and t is put $=0$, the first term in the index becomes $+\infty$ and the second term $-\infty$. As, however, the first term is proportional to the logarithm of the second, it must be very much smaller. The whole index can therefore be taken as $-\infty$, and consequently $v=0$ when $t=0$, for $r\neq 0$.

The expression (16) gives therefore the distribution of temperature that would be produced if a quantity of heat Q were suddenly created at the origin at the time $t=0$. In other words, it is the distribution of temperature due to an instantaneous point source of strength Q at the origin at time $t=0$.

Let us consider the expression (16) in more detail. For any given value of t, v diminishes as r increases, and is always 0 when $r=\infty$.

$$\frac{\partial v}{\partial t} = \frac{Q}{8c\rho(\pi\kappa t)^{\frac{3}{2}}}e^{-\frac{r^2}{4\kappa t}}\left(\frac{r^2}{4\kappa t^2} - \frac{3}{2t}\right),$$

and is zero when $t=\infty$ and when $r^2 = 6\kappa t$. The first value obviously gives a minimum since it makes v zero. The second gives a maximum, as may be shown also by forming the second derivative $\frac{\partial^2 v}{\partial t^2}$. Consider the sphere of radius b with its centre at the origin. Its surface temperature is zero when $t=0$, it increases until $t = b^2/6\kappa$, and then decreases and becomes zero again when $t=\infty$.

A source of strength $-Q$ is called a sink of strength Q.

If at any point heat is generated gradually at such a rate as to give Q units of heat per unit of time, then the point is said to be a permanent point source of strength Q.

We can easily derive the temperature distribution for a permanent point source of strength Q at the origin, for we have only to multiply

(16) by dt and integrate from 0 to ∞. An infinite time is necessary for the permanent state to be established. Then

$$v = \frac{\mathrm{Q}}{8c\rho(\pi\kappa)^{\frac{3}{2}}}\int_0^\infty e^{-\frac{r^2}{4\kappa t}}\frac{dt}{t^{\frac{3}{2}}}.$$

Substitute z for $\dfrac{r}{2\sqrt{\kappa t}}$.

Then $$dz = -\frac{r\,dt}{4\kappa^{\frac{1}{2}}t^{\frac{3}{2}}} \quad \text{and} \quad v = \frac{-\mathrm{Q}}{2c\rho\pi^{\frac{3}{2}}\kappa r}\int_\infty^0 e^{-z^2}dz = \frac{\mathrm{Q}}{4\pi k r}, \quad \text{.........(17)}$$

since $$\int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad k = c\rho\kappa.$$

This result might have been obtained from § 73. For there

$$v = (\mathrm{V}_a - \mathrm{V}_b)\frac{ab}{b-a}\frac{1}{r} + \frac{\mathrm{V}_b b - \mathrm{V}_a a}{b-a},$$

and the quantity of heat that flows across any isothermal in the unit of time, here denoted by Q, is given there by

$$4\pi k(\mathrm{V}_a - \mathrm{V}_b)\frac{ab}{(b-a)}.$$

Hence $$v = \frac{\mathrm{Q}}{4\pi k r} + \frac{\mathrm{V}_b b - \mathrm{V}_a a}{b-a},$$

which agrees with (17) to a constant term. In § 73, it is to be remembered, the temperature on two concentric spheres was arbitrarily defined. In this section the temperature is defined, so that it is zero at infinity.

§ 89. **Plane source of heat.**

It may be shown in the same way as in § 88 that

$$v = \frac{\mathrm{Q}}{2c\rho\sqrt{\pi\kappa t}}e^{-\frac{(\xi - x)^2}{4\kappa t}}$$

is the distribution of temperature due to an instantaneous plane source of strength Q given by $x = \xi$. The quantity Q is in this case the amount of heat instantaneously generated per unit area of the given plane. It is, however, more instructive to derive the result from the expression for the temperature in the doubly-infinite solid, namely,

$$v = \frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{+\infty} f(\xi)e^{-\frac{(\xi - x)^2}{4\kappa t}}d\xi. \quad \text{.....................(18)}$$

Suppose that a quantity of heat Q is suddenly given to the space bounded by the two planes ξ and $\xi + \delta\xi$ for every unit area of the planes. Then the temperature of this space becomes

$$\frac{\mathrm{Q}}{c\rho\delta\xi},$$

and the temperature of the rest of the solid is of course zero, *i.e.* $f(\xi)=\dfrac{Q}{c\rho\delta\xi}$ between ξ and $\xi+\delta\xi$, and is elsewhere 0. Substituting in the formula, we obtain

$$v=\frac{1}{2\sqrt{\pi\kappa t}}\int_{\xi}^{\xi+\delta\xi}\frac{Q}{c\rho\delta\xi}e^{-\frac{(\xi-x)^2}{4\kappa t}}d\xi=\frac{Q}{2c\rho\sqrt{\pi\kappa t}}e^{-\frac{(\xi-x)^2}{4\kappa t}},$$

when $\delta\xi$ is made infinitely small.

We can thus regard (18) as the solution for an instantaneous heat source of strength $c\rho f(\xi)\delta\xi$ per unit area on every plane ξ.

§ 90. **Doublets.**

Suppose that we have an instantaneous point source of strength Q and an instantaneous point sink of strength Q situated respectively at

$$x=\frac{l}{2},\ y=0,\ z=0 \quad \text{and} \quad x=-\frac{l}{2},\ y=0,\ z=0,$$

where l is small, then the source and sink together are said to constitute a doublet of strength Ql.

The temperature distribution due to the doublet is by (16) obviously given by

$$v=\frac{Q}{8c\rho(\pi\kappa t)^{\frac{3}{2}}}e^{-\frac{\left(x-\frac{l}{2}\right)^2+y^2+z^2}{4\kappa t}}-\frac{Q}{8c\rho(\pi\kappa t)^{\frac{3}{2}}}e^{-\frac{\left(x+\frac{l}{2}\right)^2+y^2+z^2}{4\kappa t}}$$

$$=\frac{Q}{8c\rho(\pi\kappa t)^{\frac{3}{2}}}e^{-\frac{(x^2+y^2+z^2)}{4\kappa t}}\left[e^{\frac{lx-\frac{l^2}{4}}{4\kappa t}}-e^{\frac{-lx-\frac{l^2}{4}}{4\kappa t}}\right].$$

Since l is small, the square bracket becomes

$$\left\{1+\frac{\left(lx-\frac{l^2}{4}\right)}{4\kappa t}\dots\right\}-\left\{1-\frac{\left(lx+\frac{l^2}{4}\right)}{4\kappa t}\dots\right\}=\frac{lx}{2\kappa t}.$$

Hence $$v=\frac{Qlx}{16c\rho\pi^{\frac{3}{2}}\kappa^{\frac{5}{2}}t^{\frac{5}{2}}}e^{-\frac{r^2}{4\kappa t}}.$$

This expression might have been derived from the solution for a point source by differentiating with respect to x. Since the differential equation for the conduction of heat is linear in v, we can differentiate any particular solution any number of times, and the result will still be a solution.

§ 91. **Two and three-dimensional Fourier series and integrals.** Consider the following problem:

$$(1)\quad \frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right);$$

$$(2)\quad \left.\begin{array}{ll} v=0 \text{ for } x=0, & v=0 \text{ for } x=a, \\ v=0 \text{ for } y=0, & v=0 \text{ for } y=b, \end{array}\right\} \text{ for all } t;$$

$$(3)\quad v=f(x,y) \text{ for } t=0, \quad v\neq\infty \text{ for } t=\infty.$$

The boundary conditions suggest sines; therefore write

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

for the part of the particular solution into which x and y enter, n and m being integers, and try $e^{\alpha t}$ as a time factor. By substituting in (1), we find that

$$\alpha=-\kappa\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}.$$

Hence the particular solution is

$$v=e^{-\kappa\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}t}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

This satisfies the condition for $t=\infty$. We have now only to satisfy the condition for $t=0$.

When we consider the latter, we find a new feature, the temperature being given as an arbitrary function of x and y instead of x only. Also the series which we obtain by multiplying all the particular integrals by constants and adding, namely,

$$\Sigma\Sigma b_{mn}e^{-\kappa\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}t}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

is a doubly infinite one, *i.e.* for every value of m there are an infinite number of values of n and vice versa. The question therefore arises, whether for $t=0$ we can represent such a function by such a series?

Let us first of all regard y as constant. Then $f(x, y)$ can be expanded in terms of $\sin\frac{m\pi x}{a}$ by the half range sine series. The expansion is

$$f(x,y)_{y=\text{const.}}=\Sigma b_m\sin\frac{m\pi x}{a},$$

where $$b_m=\frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,dx.$$

After the integration with respect to x is performed and the limits

substituted, regard b_m as a function of y and let it be expanded in terms of $\sin\frac{n\pi y}{b}$ by the half range sine series. Thus

$$b_m = \Sigma b_{mn} \sin\frac{n\pi y}{b},$$

where
$$b_{mn} = \frac{4}{ab}\int_0^b dy \int_0^a f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx.$$

Substitute this value of b_m in the original series and

$$f(x, y) = \Sigma\Sigma b_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

where b_{mn} has the value given three lines above.

We have therefore shown the possibility of expanding $f(x, y)$ from $x=0$ to $x=a$ and from $y=0$ to $y=b$ by such a doubly infinite series, and the solution of the problem is

$$v = \Sigma\Sigma b_{mn} e^{-\kappa\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

where b_{mn} has the above mentioned value.

Similarly it can be shown that $f(x, y, z)$ can be expanded within the range $x=0$ to $x=a$, $y=0$ to $y=b$, $z=0$ to $z=c$ by the triply infinite series

$$\Sigma\Sigma\Sigma b_{mnp} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c},$$

where
$$b_{mnp} = \frac{8}{abc}\int_0^c dz \int_0^b dy \int_0^a f(x, y, z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c}\, dx.$$

We can also expand $f(x, y)$ and $f(x, y, z)$ in terms of the products of cosines or of cosines and sines.

If the range is infinite, these two and three-dimensional series become two and three-dimensional integrals. The most formidable in appearance, the three-dimensional doubly infinite one, is as follows:

$$\frac{1}{\pi^3}\int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} f(\xi, \eta, \zeta)$$
$$\cos\alpha(\xi - x) \cos\beta(\eta - y) \cos\gamma(\zeta - z)\, d\zeta.$$

It represents $f(x, y, z)$ throughout all space.

EXAMPLES.

1. Prove that
$$v = \frac{Q}{4\pi c\rho\kappa t} e^{-\frac{r^2}{4\kappa t}}$$

represents an instantaneous point source of strength Q in an infinite thin plate, the surfaces of which are impervious to heat. Also show that the maximum value of v at a point distant r from the source is

$$\frac{Q}{\pi c\rho r^2 e}.$$

After what time is this maximum attained?

2. A quantity of heat Q is imparted at a given instant to an infinite uniform solid at a point O. Find the radius of the sphere which separates the region for which the temperature is rising from the region for which the temperature is falling. Show that its rate of increase is inversely proportional to its magnitude.

3. Two semi-infinite solids of the same material, bounded by the plane $x=0$, are initially at temperatures uniform throughout, one at temperature v_0, the other at temperature $-v_0$. If conduction takes place across the boundary, find the temperature and gradient of temperature at any subsequent time, for any point in either.
Taking the foot and year as units of length and time, and the value of the diffusivity as 400 and of v_0 as 10,000° F., find the gradient at the surface after 200,000 years.

4. Two uniform thin bars of the same material and cross-section are infinitely long in one direction. One is throughout at temperature V and the other at temperature zero, when they are put in contact end to end. Find on the supposition of zero lateral loss of heat the temperature at any point in either after the lapse of any interval of time.

5. A bar of uniform cross-section is covered with impermeable varnish and extends from the point $x=0$ to infinity. The bar being throughout at temperature zero, the extremity is brought at time $t=0$ to temperature v_0 and kept so ever after. Find the distribution of temperature in the bar at any subsequent time t, and verify that your expression gives the obvious solution for $t=\infty$.

6. A rectangular plate bounded by the lines $x=0$, $y=0$, $x=a$, $y=b$ has an initial distribution of temperature given by

$$v=\mathrm{A}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

The edges are kept constantly at zero temperature and the plane faces are impermeable to heat. Find the temperature at any point and time, and show that very close to any corner of the plate the lines of equal temperature and flow of heat are orthogonal systems of rectangular hyperbolas.
Show that the heat lost by the plate across the edges up to time t is

$$\frac{4s\mathrm{A}ab}{\pi^2}\left\{1-e^{-\kappa\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\pi^2 t}\right\},$$

where s is the thermal capacity of the plate per unit area.

7. If the temperature of an infinite solid has different uniform values V, V′ on opposite sides of a given plane, prove that at any subsequent time the temperature is given by the expression

$$\frac{\mathrm{V}+\mathrm{V}'}{2}+\frac{\mathrm{V}-\mathrm{V}'}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\kappa t}}}e^{-\lambda^2}d\lambda,$$

x being measured from the plane towards the side where the temperature was initially V.

8. A conducting sphere initially at zero temperature has its surface kept at a constant temperature c for a given time, after which it is kept at zero. Find the temperature at any time in the second stage.

9. If the surface of a rectangular parallelopiped is kept at the temperature zero and the initial temperatures of all points of the parallelopiped are given, then for any point of the parallelopiped

$$v=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\sum_{p=1}^{p=\infty}\mathsf{B}_{m,n,p}\,e^{-\kappa\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{p^2}{c^2}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c},$$

where $$\mathsf{B}_{m,n,p}=\frac{8}{abc}\int_0^a dx\int_0^b dy\int_0^c f(x,y,z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c}dz.$$

CHAPTER IV.

WAVE MOTION.

§ 92. **Transverse vibrations of a stretched string.**

We suppose that the string is perfectly flexible, that it offers no resistance to bending and that it is stretched between two points by a constant stretching force T, so great that gravity can be neglected in comparison with it. Then the string is capable of executing vibrations of two kinds,

(1) transverse vibrations, in which every particle moves at right angles to the length of the string, and

(2) longitudinal vibrations, in which every particle moves parallel to the length of the string.

If the string is displaced and left to itself we have vibrations of both kinds occurring together, but the longitudinal vibrations can usually be neglected in comparison with the transverse vibrations. We shall assume, in what follows, that this is the case, that the transverse displacement is a small quantity of the first order and that the longitudinal displacement is a small quantity of the second order in comparison with the length of the string.

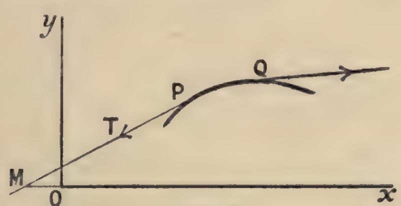

Fig. 49.

Let the string be uniform and let its mass per unit length be denoted by ρ. Take its undisturbed position as x-axis and suppose that the motion is confined to the xy-plane.

Consider the motion of an element, PQ, of length ds. Its transverse rate of change of momentum is $\rho\, ds \dfrac{\partial^2 y}{\partial t^2}$. The resultant stretching forces at P and Q act along the tangents at these points. The transverse

component of the stretching force on the element at P is $-\mathsf{T}\dfrac{\partial y}{\partial s}$, $\dfrac{\partial y}{\partial s}$ being the sine of angle P$\mathsf{M}x$ and the transverse component of the stretching force on the element at Q is $\mathsf{T}\dfrac{\partial y}{\partial s}+\dfrac{\partial}{\partial s}\left(\mathsf{T}\dfrac{\partial y}{\partial s}\right)ds$. The resultant of these two forces is a force of amount

$$\frac{\partial}{\partial s}\left(\mathsf{T}\frac{\partial y}{\partial s}\right)ds$$

parallel to Oy and in the direction of Oy. The stretching force may be regarded as constant throughout the string, also since the displacement is small we may write $\dfrac{\partial y}{\partial x}$ for $\dfrac{\partial y}{\partial s}$ in the above expression. The resultant force on the element may therefore be written $\mathsf{T}\dfrac{\partial^2 y}{\partial x^2}ds$. Equating the rate of increase of momentum to this, we obtain

$$\rho\frac{\partial^2 y}{\partial t^2}=\mathsf{T}\frac{\partial^2 y}{\partial x^2}$$

for the equation of motion of the string. If we write v^2 for T/ρ, this takes the form

$$\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x^2}.$$

§ 93. Let us change the independent variables in the above equation to x_1 and x_2, these quantities being given by

$$x_1=x-vt,\quad x_2=x+vt.$$

Then $$\frac{\partial}{\partial x}=\frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x}+\frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x}=\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}.$$

Therefore $$\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial x_1^2}+2\frac{\partial^2 y}{\partial x_1\partial x_2}+\frac{\partial^2 y}{\partial x_2^2}.$$

Also $$\frac{\partial}{\partial t}=\frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial t}+\frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial t}=-v\frac{\partial}{\partial x_1}+v\frac{\partial}{\partial x_2},$$

and consequently $$\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x_1^2}-2v^2\frac{\partial^2 y}{\partial x_1\partial x_2}+v^2\frac{\partial^2 y}{\partial x_2^2}.$$

On substituting these values, the original equation reduces to

$$\frac{\partial^2 y}{\partial x_1\partial x_2}=0.$$

The most general solution of this is obviously

$$y=f_1(x_1)+f_2(x_2).$$

Hence the most general solution of the original equation is

$$y=f_1(x-vt)+f_2(x+vt).$$

Now, if $f_1(x-vt)$ be plotted as a function of x, it is exactly the same as $f_1(x)$ in shape, but every point on it is displaced a distance vt to the right of the corresponding point in $f_1(x)$. It thus represents an irregular wave travelling towards the right with uniform velocity v, the shape of the wave at time $t=0$ being given by $y=f_1(x)$. Similarly $y=f_2(x+vt)$ represents a wave travelling towards the left with uniform velocity v. The general solution is the sum of these two waves.

Let the string have one end fixed at the origin, let the other end be a great distance off in the direction of Ox, and suppose that a wave given by $y=f(vt+x)$ is approaching the origin. At the origin the displacement must be zero; hence the reflected wave must have the form $y=-f(vt-x)$, since the sum of this and the original expression is zero at $x=0$ for all values of t. The transverse wave in a stretched string is therefore inverted by reflection.

§ 94. **Harmonic waves.**

Consider the expression

$$y=\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)=\cos 2\pi\left(\frac{t}{\tau}-\frac{x}{\lambda}\right).$$

By plotting it as a function of x for successive values of t, it may be shown to represent an infinite train of progressive harmonic waves. The waves are said to be harmonic because the displacement has the cosine form, the train is infinite as the expression gives real values for the displacement throughout the whole range $-\infty<x<+\infty$, and the waves are said to be progressive because as t increases the whole wave profile moves bodily forward in the direction of positive x. The wave-length, the distance between two successive crests at any instant, is given by λ; the period, that is the time taken by a complete wave to pass a fixed point, is given by τ.

Let us suppose that the progressive wave travelling from right to left and given by

$$y=\cos 2\pi\left(\frac{t}{\tau}+\frac{x}{\lambda}\right)$$

is reflected at the origin. Then the reflected wave must be given by

$$y=-\cos 2\pi\left(\frac{t}{\tau}-\frac{x}{\lambda}\right).$$

The resultant displacement at any point on the string due to the superposition of the two waves is given by

$$y=\cos 2\pi\left(\frac{t}{\tau}+\frac{x}{\lambda}\right)-\cos 2\pi\left(\frac{t}{\tau}-\frac{x}{\lambda}\right)=-2\sin\frac{2\pi t}{\tau}\sin\frac{2\pi x}{\lambda}.$$

By plotting this expression as a function of x for successive values of t, it may be shown to represent an infinite train of stationary harmonic waves. At the points given by $x=0,\ \frac{\lambda}{2},\ \lambda,\ \frac{3\lambda}{2},\ \ldots$, which are called nodes, the displacement is always zero. The points midway between

the nodes are called loops, and at the loops the displacement varies between $+2$ and -2.

A good example of a solitary wave, as opposed to an infinite train, is given by the expression

$$y = e^{-c(x-vt)^2},$$

which can easily be shown graphically to represent a solitary maximum moving in the direction of positive x.

§ 95. **String of length l.**

Let us suppose that the length of the string is l and that its initial displacement and velocity are given. Then the problem may be stated as follows:

$$(1)\ \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}, \text{ where } v^2 = \frac{\mathrm{T}}{\rho}; \quad (2)\ y = 0 \text{ for } x = 0,\ x = l;$$

$$(3)\ y = f(x),\ \frac{\partial y}{\partial t} = \phi(x) \text{ for } t = 0.$$

As in the case of the differential equation for the conduction of heat, we build up the solutions from sines and cosines or exponentials. We know from § 93 that the expression

$$\mathrm{P}e^{im\left(\frac{x}{v}+t\right)} + \mathrm{Q}e^{im\left(\frac{x}{v}-t\right)} + \mathrm{R}e^{-im\left(\frac{x}{v}+t\right)} + \mathrm{S}e^{-im\left(\frac{x}{v}-t\right)}$$

satisfies (1). The constants must be chosen to make it satisfy (2). If we write $\mathrm{S} = -\mathrm{P}$, $\mathrm{R} = -\mathrm{Q}$, it becomes

$$\mathrm{P}e^{imt}\left(e^{i\frac{mx}{v}} - e^{-i\frac{mx}{v}}\right) + \mathrm{Q}e^{-imt}\left(e^{i\frac{mx}{v}} - e^{-i\frac{mx}{v}}\right)$$

$$= 2i \sin\frac{mx}{v}(\mathrm{P}e^{imt} + \mathrm{Q}e^{-imt}) = \sin\frac{mx}{v}(b\cos mt + b'\sin mt),$$

b and b' being constants. This satisfies (2) if $m/v = n\pi/l$, where n is any integer.

On substituting for m and taking all the possible values of n, we obtain

$$y = \sum_{n=1}^{n=\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l} + \sum_{n=1}^{n=\infty} b'_n \sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}. \quad \ldots\ldots\ldots(1)$$

We then choose the constants b_n and b'_n so as to satisfy (3). When $t = 0$, y is represented by the half range sine series

$$\Sigma b_n \sin\frac{n\pi x}{l} = f(x)$$

and $\frac{\partial y}{\partial t}$ by the half range sine series

$$\Sigma b'_n \frac{n\pi v}{l}\sin\frac{n\pi x}{l} = \phi(x).$$

Hence, by § 64,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \quad \text{and} \quad b'_n = \frac{2}{n\pi v}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx.$$

Each of the terms in (1) represents a stationary wave, the wave-lengths being given by $2l/n$, where n is any integer. The frequency, or number of periods per second, of the fundamental note is $\frac{1}{2l}\sqrt{\frac{T}{\rho}}$ and the frequency of its harmonics, as the other terms are called, is given by the well-known formula $\frac{n}{2l}\sqrt{\frac{T}{\rho}}$.

The initial conditions, that is the values of $f(x)$ and $\phi(x)$, may be such as to make some of the constants b_n, b_n' vanish. In that case the corresponding harmonics in the note are awanting.

If the initial displacement and velocity are not confined to one plane, we resolve them into components in the xy and zx-planes. The principle of the superposition of small vibrations then enables us to treat the motion in the one plane quite independently of the motion in the other.

§ 96. **Damping.**

Vibrations are said to be damped when their amplitude decreases with time. All vibrations that occur in nature are damped. So far we have not taken account of damping, and our results are therefore strictly true only for an ideal string. We can represent the effect of damping by adding an additional term $-2k\frac{\partial y}{\partial t}$ to the right-hand side of the equation of motion for the stretched string. This term represents a force proportional to the velocity and resisting the motion. It is due to friction in the string and to loss of energy by air waves, but it is not possible to form a clear picture as to how it acts.

Let us suppose that the boundary and initial conditions are the same as in § 95, but that instead of (1) we have the equation

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} - 2k\frac{\partial y}{\partial t}. \qquad \text{.......(2)}$$

Try $e^{\pm i\alpha x + \beta t}$ as a solution. Then

$$\beta^2 + v^2\alpha^2 + 2k\beta = 0,$$

i.e. $$\beta = -k \pm \sqrt{k^2 - v^2\alpha^2} \quad \text{or} \quad -k \pm i\sqrt{v^2\alpha^2 - k^2},$$

since k is presumably small.

The typical solution is therefore $e^{-kt \pm i\alpha x \pm i\sqrt{v^2\alpha^2 - k^2}t}$. Combining expressions of this type in the same way as in the last solution, we obtain

$$e^{-kt}\sin\frac{n\pi x}{l}\left(b\cos\sqrt{\frac{n^2\pi^2v^2}{l^2} - k^2}t + b'\sin\sqrt{\frac{n^2\pi^2v^2}{l^2} - k^2}t\right)$$

as an expression satisfying the end conditions.

The period of the n^{th} component is therefore $2\pi\Big/\sqrt{\frac{n^2\pi^2v^2}{l^2}-k^2}$. It has been increased by damping. Also the overtones are no longer harmonics of the fundamental.

Let us now consider the initial conditions. When $t=0$ the second term in the above expression vanishes, and the first term takes the same value as before. But on differentiating the expression with regard to t and afterwards putting $t=0$, we obtain

$$\sin\frac{n\pi x}{l}\left(-bk+b'\sqrt{\frac{n^2\pi^2v^2}{l^2}-k^2}\right).$$

In order to get rid of the additional term, we must write the complete solution,

$$y=\sum_{n=1}^{n=\infty}b_n e^{-kt}\sin\frac{n\pi x}{l}\cos\sqrt{\frac{n^2\pi^2v^2}{l^2}-k^2}\,t$$
$$+\sum_{n=1}^{n=\infty}\left(b'_n+\frac{b_nk}{\sqrt{\frac{n^2\pi^2v^2}{l^2}-k^2}}\right)e^{-kt}\sin\frac{n\pi x}{l}\sin\sqrt{\frac{n^2\pi^2v^2}{l^2}-k^2}\,t.$$

It is obvious that this satisfies the initial conditions when the values of b_n and b'_n are given by

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \quad \text{and} \quad b'_n=\frac{2}{l\sqrt{\frac{n^2\pi^2v^2}{l^2}-k^2}}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx.$$

§ 97. **Energy of a vibrating string.**

Let us assume that the motion is undamped, that the string is of length l and that it is displaced in one plane and let go. Then, when $t=0$, $y=f(x)$ and $\frac{\partial y}{\partial t}=0$, and the solution is consequently

$$y=\sum_{n=1}^{n=\infty}b_n\sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}.$$

We shall now determine the kinetic energy of the string at time t. At that time we have

$$\frac{\partial y}{\partial t}=-\sum_{n=1}^{n=\infty}b_n\frac{n\pi v}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}.$$

Now the kinetic energy is

$$\frac{1}{2}\int_0^l \rho\dot{y}^2\,dx=\frac{\rho}{2}\Sigma b_n^{\,2}\left(\frac{n\pi v}{l}\right)^2\sin^2\frac{n\pi vt}{l}\int_0^l\sin^2\frac{n\pi x}{l}\,dx,$$

for all the integrals of the type $\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$ vanish. Also

$$\int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_0^l \left(1-\cos\frac{2n\pi x}{l}\right)dx = \frac{l}{2}.$$

Hence the kinetic energy is given by

$$\frac{m}{4}\Sigma\, b_n{}^2\left(\frac{n\pi v}{l}\right)^2 \sin^2\frac{n\pi vt}{l},$$

m being substituted for ρl, the total mass of the string. The kinetic energy is thus equal to the sum of the kinetic energies of the different modes of vibration.

In order to determine the potential energy of the string, we must find the work done in displacing it from the x-axis to the actual position occupied at time t. The force per unit of mass in the direction of the y-axis according to the equation of motion is $v^2\frac{\partial^2 y}{\partial x^2}$. Hence the work done in displacing the element through δy is given by

$$-\rho v^2\frac{\partial^2 y}{\partial x^2}\,ds\,\delta y.$$

Suppose that the string is brought to its final position in a number of steps, δy for any one step being the same fraction of the final value of y for every point on the string. Then δy is a function of x. Writing $ds = dx$, we find for the work done in any one step

$$-\rho v^2\int_0^l \frac{\partial^2 y}{\partial x^2}\,\delta y\,dx = \rho v^2\int_0^l \frac{\partial y}{\partial x}\frac{\partial\,\delta y}{\partial x}\,dx,$$

integrating by parts, δy being zero for the ends of the string. This is equal to

$$\frac{\rho v^2}{2}\int_0^l \delta\left(\frac{\partial y}{\partial x}\right)^2 dx.$$

The amount of work done in all the steps is therefore

$$\frac{\rho v^2}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{\rho v^2}{2}\int_0^l \Sigma\left(\frac{b_n n\pi}{l}\right)^2 \cos^2\frac{n\pi x}{l}\cos^2\frac{n\pi vt}{l}\,dx$$
$$= \frac{\rho v^2 l}{4}\Sigma\left(\frac{b_n n\pi}{l}\right)^2\cos^2\frac{n\pi vt}{l},$$

the product terms vanishing, as before, during the integration. Writing m for ρl, we obtain for the potential energy of the string

$$\frac{m}{4}\Sigma b_n{}^2\left(\frac{n\pi v}{l}\right)^2\cos^2\frac{n\pi vt}{l}.$$

The result might have been derived by the energy principle from the expression for the kinetic energy, since the potential energy is zero when the string is crossing its equilibrium position.

§ 98. **Longitudinal vibrations in a rod.**

The longitudinal vibrations in a rod of any section and torsional vibrations in a rod of circular section are mathematically the same as the transverse vibrations in a stretched string. We shall proceed now to deal with these vibrations, neglecting gravity in each case.

Let us suppose we have a uniform rod of density ρ and cross-sectional area a. Take the x-axis in the direction of its length, and let E be the Young's modulus* of the material of which it is composed.

Consider the element of the rod bounded by the planes A and B, which are given respectively by x and $x+dx$. Its mass is $\rho a\,dx$. Let a longitudinal wave pass down the rod. Then all its particles will be displaced in the direction of $\mathsf{O}x$. Let ξ denote the displacement of the particles that were originally at x. The elongation per unit length at A is $\frac{\partial\xi}{\partial x}$; hence the total force exerted on the element from right to left across the plane A is $\mathsf{E}a\frac{\partial\xi}{\partial x}$. The total force exerted on the element from left to right across the plane B is $\mathsf{E}a\left(\frac{\partial\xi}{\partial x}+\frac{\partial^2\xi}{\partial x^2}dx\right)$. Consequently the resultant force on the element in the x direction is $\mathsf{E}a\,dx\frac{\partial^2\xi}{\partial x^2}$. The acceleration of the element in the same direction is $\frac{\partial^2\xi}{\partial t^2}$. The equation of motion is therefore

$$\rho a\,dx\frac{\partial^2\xi}{\partial t^2}=\mathsf{E}a\,dx\frac{\partial^2\xi}{\partial x^2} \quad\text{or}\quad \frac{\partial^2\xi}{\partial t^2}=\frac{\mathsf{E}}{\rho}\frac{\partial^2\xi}{\partial x^2}.$$

The velocity of the wave is given by $\sqrt{\mathsf{E}/\rho}$. If one end or point of the bar is fixed, then ξ must be zero for that end or point. At a free end $\frac{\partial\xi}{\partial x}=0$, for there is no force exerted across a free end.

The periods of the different overtones and the case of given initial conditions can be worked out in the same way as for the transverse vibrations of a stretched string.

It should be noticed that, just as in the case of a stretched wire, where the rod is stretched, it suffers lateral contraction, and that where it is compressed longitudinally, it suffers lateral expansion. The cross-sectional area a is thus not constant, but the error introduced by considering it constant can be neglected.

§ 99. **Torsional vibrations in a right circular cylinder.**

Let a be the radius of the cylinder, ρ the density and n the rigidity modulus of the material of which it is composed. Take the axis of the cylinder as axis of x, let the cylinder be vertical and take the origin in its upper surface. Let the upper surface be fixed and let

* If a wire of length L and cross-sectional area a is stretched a small distance l by a force F, then its Young's modulus

$$=\frac{\text{stretching force per unit area}}{\text{elongation per unit length}}=\frac{\mathsf{FL}}{al}.$$

the cylinder be twisted about its axis. Denote by θ the angle through which the plane defined by x is twisted.

To find the twisting couple in any section of the cylinder, consider the slice bounded by x and $x+dx$. Divide it into rings by drawing coaxal cylinders with Ox as axis, and consider the ring bounded by r and $r+dr$. Its upper surface is twisted through an angle θ and its lower surface through an angle $\theta+\frac{\partial\theta}{\partial x}dx$. If we divide the ring into elements by drawing planes through the axis of the cylinder and consider any one of these elements, when the cylinder is twisted, its lower surface is displaced a distance $r\frac{\partial\theta}{\partial x}dx$ further round than its upper surface. Each element of the ring therefore (cf. fig. 51) suffers shearing strain * ϕ given by

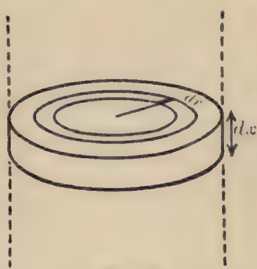

FIG. 50.

$$\phi=r\frac{\partial\theta}{\partial x},$$

the expression for ϕ being obtained by dividing the displacement $r\frac{\partial\theta}{\partial x}dx$ by dx, the distance between the two planes.

The tangential force per unit area on the upper surface of the element is $nr\frac{\partial\theta}{\partial x}$. The moment of this about the axis is $nr^2\frac{\partial\theta}{\partial x}$. The total moment about the axis of all the tangential forces on the upper surface of the ring is therefore $2\pi nr^3\frac{\partial\theta}{\partial x}dr$ and the resultant couple exerted on the section is

FIG. 51.

$$2\pi n\frac{\partial\theta}{\partial x}\int_0^a r^3dr=\pi n\frac{a^4}{2}\frac{\partial\theta}{\partial x}.$$

* When a cube is deformed in the manner illustrated in the diagram, it is said to suffer shearing strain. Note that the new position of the upper face, A′B′F′E′, is still in the same plane as the old. The amount of the strain is measured by AA′/AD or the angle ϕ, since ϕ is small.

Such a strain is produced by a system of equal tangential forces acting on A′F′, CF′ and the two opposite faces in the directions indicated by the arrows. Let P be the amount of each of these forces per unit area of the face on which it acts. Then n, the modulus of rigidity of the material of which the cube is composed, is defined by

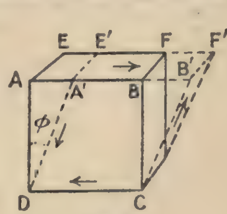

FIG. 52.

$$n=\frac{P}{\phi}.$$

Let us now consider the motion of an element bounded by the sections x and $x+dx$. Its mass is $\rho\pi a^2 dx$ and its rate of increase of angular momentum about the x-axis is

$$\rho\pi a^2 dx \frac{a^2}{2} \frac{\partial^2\theta}{\partial t^2},$$

$a^2/2$ being the square of its radius of gyration. The twisting couple on the upper surface of this element is $\pi n \frac{a^4}{2} \frac{\partial\theta}{\partial x}$ and it is in the negative direction about Ox. The couple on its lower surface is $\frac{\pi n a^4}{2}\left(\frac{\partial\theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}dx\right)$ in the positive direction. The resultant couple is therefore

$$\frac{\pi n a^4}{2} \frac{\partial^2\theta}{\partial x^2} dx.$$

The equation of motion of the element is thus

$$\frac{\rho\pi a^4}{2} \frac{\partial^2\theta}{\partial t^2} dx = \frac{\pi n a^4}{2} \frac{\partial^2\theta}{\partial x^2} dx \quad \text{or} \quad \frac{\partial^2\theta}{\partial t^2} = \frac{n}{\rho} \frac{\partial^2\theta}{\partial x^2}.$$

Torsional waves are therefore propagated along the cylinder with a velocity $\sqrt{n/\rho}$. At a fixed end of the cylinder $\theta=0$; at a free end $\frac{\partial\theta}{\partial x}=0$, since the torsional couple there must be zero.

Let us suppose that the cylinder is a thin wire of length l, the upper end of which is fixed and to the lower end of which a heavy cylindrical vibrator is attached. Let M be the mass and k^2 the square of the radius of gyration of this vibrator. In this case the condition to be satisfied at the lower end of the wire is that the torsional couple there should be equal to the rate of increase of angular momentum of the vibrator. That is, for $x=l$,

$$-\frac{\pi n a^4}{2} \frac{\partial\theta}{\partial x} = \mathsf{M}k^2 \frac{\partial^2\theta}{\partial t^2}.$$

Assuming $\theta = (\mathsf{A}\cos mx + \mathsf{B}\sin mx)\cos mvt$ as a solution, we find from the condition for $x=0$ that A must be zero. From the condition for $x=l$,

$$-\frac{\pi n a^4}{2} \mathsf{B} m \cos ml = -\mathsf{M}k^2\mathsf{B}(mv)^2 \sin ml \quad \text{or} \quad \tan ml = \frac{\pi n a^4}{2v^2\mathsf{M}k^2 m}.$$

This equation gives m.

If Mk^2 be very great, as is the case in the usual experimental method for determining n, $\tan ml$ is small, and ml can be written for it. Then

$$m^2 = \frac{\pi n a^4}{2v^2\mathsf{M}k^2 l}. \quad \text{.................................(3)}$$

Also, owing to ml being small, the solution can be written

$$\theta = \mathsf{B}mx \cos mvt;$$

that is, the angle of twist of any cross-section is proportional to its distance from the upper end of the wire. The period of vibration, τ, is equal to $2\pi/mv$. Hence, substituting in (3), we obtain

$$n=\frac{8\pi Mk^2l}{a^4\tau^2},$$

the usual formula for the determination of the rigidity modulus.

EXAMPLES.

1. Find the form at time t of a vibrating string of length l, whose ends are fixed and which is initially displaced into an isosceles triangle. The string is vibrating transversely, is under constant stretching force and starts from rest.

2. A portion of an infinite isotropic solid is contained between two parallel planes at a distance l apart. It is fixed at these planes and vibrates in a fixed direction parallel to them at points between. Establish the differential equation for such vibrations, and give the complete solution for the problem in question.

3. To the bottom of the vibrator of one torsional pendulum is fastened rigidly a wire which carries at its other end another torsional vibrator. The two wires are vertical and collinear, and the two vibrators execute small vibrations in horizontal planes. Supposing that the usual condition for uniform twist (which is to be stated) is satisfied for each pendulum, find completely the resulting motion when each vibrator receives initially a given displacement but no velocity.

4. The longitudinal displacements of a vertical steel rod, fixed at both ends, are given by $\xi=a\sin\frac{3\pi x}{l}$, where l is the length of the rod and x is measured from an end. Find numerically the maximum values of the terms in the differential equation and the value of the term due to gravity, which is neglected, given that $a=\cdot01$ mm., $l=100$ cms., $\rho=7\cdot7$ gms./c.c. and $E=3\ 10^9$ gms./sq. cm.

5. A transversely vibrating string of length l is stretched between two points A and B. The initial displacement of each point of the string is zero, the initial velocity at a distance x from A is $kx(l-x)$. Find the form of the string at any subsequent time.

6. A torsional vibrator is used to determine n, the rigidity modulus of a thin wire. Derive an expression for n, retaining the first two terms in the expansion for $\tan ml$ (cf. § 99). Hence find the error caused by the assumption of uniform shear in determining the rigidity modulus of a copper wire 2 metres long, the moment of inertia of the vibrator being 30,000 C.G.S. units and the period being 7 secs. Take $4\cdot5\ 10^8$ gms./sq. cm. for the rigidity modulus of copper.

7. A string of length $l+l'$ is stretched with tension T between two fixed points. The linear densities of the lengths l, l' are ρ, ρ' respectively; prove that the periods τ of transverse vibrations are given by

$$\rho'^{\frac{1}{2}}\tan\left(2\pi l\rho^{\frac{1}{2}}/\tau T^{\frac{1}{2}}\right)=\rho^{\frac{1}{2}}\tan\left(2\pi l'\rho'^{\frac{1}{2}}/\tau T^{\frac{1}{2}}\right).$$

8. If a uniform horizontal bar, both of whose ends are fixed, be displaced horizontally, so that one half is uniformly extended and the other half is uniformly compressed, prove that the displacement at time t of any particle whose abscissa is x, is

$$\left(\frac{8\mathrm{A}}{\pi^2}\right)\sum\frac{1}{(2m+1)^2}\cos\frac{(2m+1)\pi vt}{2l}\cos\frac{(2m+1)\pi x}{2l},$$

where $2l$ is the length of the bar, the middle of which is the origin, and A is the initial displacement of that point.

9. An elastic rod of length l lies on a smooth plane, and is longitudinally compressed between two pegs at a distance l' apart. One peg is suddenly removed; prove that the rod leaves the other peg just as it reaches its natural state, and then proceeds with a velocity equal to $\mathrm{V}(l-l')/l$, where V is the velocity of propagation of a longitudinal wave in the rod.

§ 100. Tidal waves.

We pass now to another case of wave-motion represented by the same differential equation, namely the case of "tidal" or "long" waves in an incompressible liquid of uniform depth h. The waves which occur on the surface of a liquid owe their propagation to two causes, surface tension in the liquid-air surface and gravity. If the wave-length is small, less than two-thirds of an inch or thereabouts, the influence of surface tension preponderates and the waves are called ripples. We shall only consider waves the wave-length of which is so great that surface tension can be neglected.

Tidal waves or long waves are a particular case of gravity waves characterised by a simpler mathematical treatment. Their distinguishing feature is, as their name implies, that the vertical displacement of the surface must be small in comparison with the wave-length.

Let the bottom of the liquid be given by $y=0$. Measure y positive upwards. Let the free surface in the xy-plane be given by $y=h+\eta$, h being the depth in the undisturbed state, and let p_0 be the pressure on the free surface. The liquid is supposed to be bounded in the z direction by fixed planes parallel to the xy-plane. The distance of these planes apart is immaterial, for they are perfectly smooth and the liquid slips along them without experiencing any frictional resistance. For simplicity we shall suppose that they are unit distance apart. The problem is then that of long waves in a uniform canal of breadth unity and depth h.

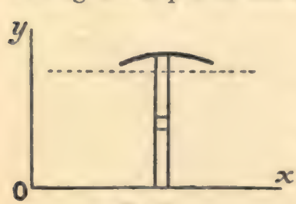

FIG. 53.

We shall make the fundamental assumption that the pressure at the point x, y is given by

$$p=p_0+g\rho(h+\eta-y). \qquad \text{.........................(4)}$$

The validity of this will be discussed later.

Consider the vertical strip which is bounded by x and $x+dx$ before the motion begins. Its volume is $h\,dx$. At a given instant after the

motion is started, let the elevation of the surface at the top of the strip be η and let the planes bounding it have moved to $x+\xi$ and $x+dx+\xi+\frac{\partial\xi}{\partial x}dx$. The volume of the strip is then $(h+\eta)\left(dx+\frac{\partial\xi}{\partial x}dx\right)$. Since the liquid is incompressible, the volume of the strip does not vary with the time. Hence

$$h\,dx=(h+\eta)\left(dx+\frac{\partial\xi}{\partial x}dx\right),$$

which gives

$$\eta+h\frac{\partial\xi}{\partial x}=0 \quad\text{.................................(5)}$$

on neglecting the second order term.

Let us now consider the equation of motion of the strip. Its rate of increase of momentum is

$$\rho h\,dx\frac{\partial^2\xi}{\partial t^2}.$$

To find the resultant force on the strip divide it into elements by planes parallel to the bottom, and consider one of these elements of height dy. The force on it towards the right is $p\,dy$ and the force towards the left $\left(p+\frac{\partial p}{\partial x}dx\right)dy$. The resultant force on the element is consequently $-\frac{\partial p}{\partial x}dx\,dy$ or, from (4), $-g\rho\frac{\partial\eta}{\partial x}dx\,dy$, since η is the only quantity in the expression for p that varies with x. The resultant force on the element is independent of y; hence all the elements of the strip must move with the same acceleration, a fact which we have already tacitly assumed, and the resultant force on the whole strip is

$$-g\rho h\frac{\partial\eta}{\partial x}dx.$$

The equation of motion of the strip is therefore

$$\rho h\,dx\frac{\partial^2\xi}{\partial t^2}=-g\rho h\frac{\partial\eta}{\partial x}dx. \quad\text{........................(6)}$$

But from the equation of continuity, (5),

$$\frac{\partial\eta}{\partial x}+h\frac{\partial^2\xi}{\partial x^2}=0.$$

Substituting this in (6) and cancelling out the common factor $\rho h\,dx$, we obtain finally

$$\frac{\partial^2\xi}{\partial t^2}=gh\frac{\partial^2\xi}{\partial x^2}, \quad\text{..............................(7)}$$

the equation of motion for the propagation of long waves.

The velocity of the wave is given by $v=\sqrt{gh}$. Assume the solution

$$\xi=A\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right). \quad\text{..............................(8)}$$

Then η, the elevation of the surface, is given by

$$\eta = -h\frac{\partial \xi}{\partial x} = -\mathsf{A}\frac{2\pi h}{\lambda}\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right).$$

If we extend the definition of η to include not only the elevation of the surface but the vertical displacement of the particle originally at any point x, y, then it is clear from the mode of deriving the principle of continuity, that η in this wider meaning is given by

$$\eta = -y\frac{\partial \xi}{\partial x} = -\mathsf{A}\frac{2\pi y}{\lambda}\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right). \quad \text{.................}(9)$$

Squaring (8) and (9) and eliminating $\left(t-\frac{x}{v}\right)$, we then obtain

$$\xi^2 + \frac{\eta^2}{\left(\frac{2\pi y}{\lambda}\right)^2} = \mathsf{A}^2,$$

the equation of an ellipse with its long axis horizontal (cf. § 101). As the wave passes, the particles of the liquid describe ellipses about their equilibrium positions as centres, the ellipses becoming thinner as we descend into the liquid and degenerating into straight lines at the bottom.

§ 101. **Condition for long waves.**

The equation of vertical motion of the liquid in the canal in the preceding section is

$$\rho\frac{dv}{dt} = -\frac{\partial p}{\partial y} - g\rho. \quad \text{..........................}(10)$$

Put the term on the left equal to zero and then integrate the equation on the assumption that the pressure has the value p_0 on the surface, that is, for $y = h + \eta$. We obtain then

$$p = p_0 + g\rho(h + \eta - y).$$

The fundamental assumption which we made in § 100 is therefore equivalent to neglecting the vertical acceleration of the particles.

To examine under what conditions the vertical acceleration may be neglected, integrate equation (10) again, retaining the left-hand term. We obtain

$$\int_c^y \rho\frac{dv}{dt}dy = -p - g\rho y, \quad \text{........................}(11)$$

the constant of integration being included in the lower limit of the integral. Substituting the condition that $p = p_0$ for $y = h + \eta$, we find

$$\int_c^{h+\eta} \rho\frac{dv}{dt}dy = -p_0 - g\rho(h+\eta),$$

and subtracting this equation from (11),

$$\int_c^y \rho\frac{dv}{dt}dy - \int_c^{h+\eta}\rho\frac{dv}{dt}dy = -p + p_0 + g\rho(h+\eta-y),$$

$$p = p_0 + g\rho(h+\eta-y) - \rho\int_{h+\eta}^y \frac{dv}{dt}dy.$$

The only terms in p concerned in the wave propagation are the variable terms $g\rho\eta$ and the last term, and in order that the waves may be long, it is necessary that the term neglected be small in comparison with the term retained, *i.e.* that $\int_{h+\eta}^{y}\frac{dv}{dt}dy$ be small in comparison with $g\eta$.

Let β be the maximum value of the vertical acceleration. Then $h\beta$ is the maximum value of the integral. Also β is of the order η/τ^2, τ being the period. Hence, if we have $h\beta$ small in comparison with $g\beta\tau^2$, our approximation is justified. Since $\lambda=\sqrt{gh}\,\tau$, this condition is equivalent to h^2/λ^2, small in comparison with 1.

§ 102. **Stationary waves in a rectangular trough.**

If a rectangular trough, the length of which is much greater than the breadth and depth, be filled with water, stationary long waves can be started by raising one end of the trough a small distance, holding it until the surface is still and then dropping it sharply. The condition to be satisfied at the ends of the trough is of course $\xi=0$, and the typical solution is $\xi=\sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}$, l being the length of the trough. The disturbance is of course not simple, consisting as it does of a great number of harmonics superimposed, but the fundamental vibration is usually predominant and persists longer. Its period can thus easily be taken with a stop watch and compared with the theoretical value $2l/\sqrt{gh}$.

§ 103. **Effect of an arbitrary initial disturbance.**

Let us suppose that the canal is unlimited in the direction of $+x$ and $-x$ and that the velocity and elevation are given initially, *i.e.*

$$\left.\begin{aligned}\frac{\partial\xi}{\partial t}&=v\phi(x)\\ \eta&=h\psi(x)\end{aligned}\right\}\text{ for } t=0.$$

Assuming the solution

$$\xi=f_1(x-vt)+f_2(x+vt),$$

we have

$$\frac{\partial\xi}{\partial t}=v\{-f'_1(x-vt)+f'_2(x+vt)\},$$

$$\eta=-h\frac{\partial\xi}{\partial x}=-h\{f'_1(x-vt)+f'_2(x+vt\}.$$

For $t=0$, therefore,

$$\phi(x)=-f'_1(x)+f'_2(x),$$
$$\psi(x)=-f'_1(x)-f'_2(x),$$

and consequently

$$f'_1(x)=-\tfrac{1}{2}\{\phi(x)+\psi(x)\},$$
$$f'_2(x)=\tfrac{1}{2}\{\phi(x)-\psi(x)\}.$$

The elevation at all subsequent times is thus given by

$$\eta=\frac{h}{2}\{\phi(x-vt)+\psi(x-vt)\}-\frac{h}{2}\{\phi(x+vt)-\psi(x+vt)\}.$$

This expression represents two waves of different shape moving in opposite directions with the same velocity. If $\frac{\partial\xi}{\partial t}=0$ for $t=0$, $\phi(x)=0$, and the two waves have the same shape. If $\phi(x)=\pm\psi(x)$, *i.e.* if for $t=0$, $\frac{\partial\xi}{\partial t}=\pm\frac{v\eta}{h}$, there is a wave only in one direction.

§ 104. Assume $\xi=f(x-vt)$. The displacement of a surface particle is given by

$$\eta=-h\frac{\partial\xi}{\partial x}=-hf'(x-vt).$$

But $$\frac{\partial\xi}{\partial t}=-vf'(x-vt).$$

Therefore $$\frac{\partial\xi}{\partial t}=+\frac{v\eta}{h}.$$

When η is positive, $\frac{\partial\xi}{\partial t}$ is positive. A wave of elevation thus moves the particles forward and a wave of depression moves them back.

§ 105. **Energy of a harmonic long wave.**

As in § 100, let the wave be given by $\xi=\mathsf{A}\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)$. We shall now determine the kinetic energy and the potential energy in one complete wave-length in a canal of unit breadth and depth h.

Consider the strip bounded by x and $x+dx$; its mass is $\rho h\,dx$ and its kinetic energy

$$\frac{1}{2}\rho h\,dx\left(\frac{\partial\xi}{\partial t}\right)^2 \quad\text{or}\quad \frac{1}{2}\rho h\,dx\left(\frac{2\pi\mathsf{A}}{\tau}\right)^2\sin^2\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right).$$

To find the kinetic energy in one complete wave-length, we must integrate this with regard to x through any range λ. The position of the range is immaterial. The result is

$$\rho h\left(\frac{\pi\mathsf{A}}{\tau}\right)^2\lambda,$$

which becomes $\frac{g\rho h^2}{\lambda}(\pi\mathsf{A})^2$ on using the relation $\lambda^2=gh\tau^2$. The kinetic energy is thus proportional to the square of the amplitude.

To obtain the potential energy, consider the accompanying diagram. The potential energy of the part GHEDCBO is unaltered by the wave; the effect of the wave is to lift the portion EDCP′ to GHEF. Divide these portions into elements. Then to an element at P of mass $\rho\eta\,dx$ will correspond an equal element at P′. The work done in lifting the one element into the position of the other is equal to $g\rho\eta^2dx$, η being the vertical distance between the centroids. If we take the original

value of the potential energy as zero, the potential energy in one complete wave-length is given by

$$\int g\rho\eta^2 dx,$$

the integration being taken through half a wave-length. As before, the particular position of the range of integration does not matter.

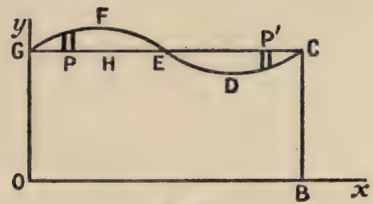

FIG. 54.

Now $$\eta = -\mathsf{A}\frac{2\pi h}{\lambda}\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right).$$

Hence the integral is

$$\int_0^{\frac{\lambda}{2}} g\rho\left(\frac{2\pi h\mathsf{A}}{\lambda}\right)^2 \sin^2\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)dx = \frac{g\rho h^2}{\lambda}(\pi\mathsf{A})^2.$$

The potential energy of the wave is therefore equal to its kinetic energy.

Now calculate the work done by the pressure in any plane normal to Ox in one period. Let $p\,dy$ be the thrust on an element of the plane. Then $\frac{\partial\xi}{\partial t}$ gives the velocity with which the point of application of this thrust is moving, $\int_0^h p\frac{\partial\xi}{\partial t}dy$ gives the rate at which work is being done in the plane and $\int_0^\tau dt\int_0^h p\frac{\partial\xi}{\partial t}dy$ gives the total work done in one period. As $\frac{\partial\xi}{\partial t}$ is as often positive as negative, the constant part of p contributes nothing to the integral, the direction in which it does work constantly altering. We can therefore substitute $g\rho\eta$ for p. The integrand is then independent of y, and the integral becomes

$$\int_0^\tau g\rho h\eta\frac{\partial\xi}{\partial t}dt = g\rho h^2\left(\frac{2\pi\mathsf{A}}{\lambda}\right)\left(\frac{2\pi\mathsf{A}}{\tau}\right)\int_0^\tau \sin^2\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)dt = \frac{2g\rho h^2}{\lambda}(\pi\mathsf{A})^2.$$

The particular time at which the period is taken is immaterial, as may be seen by substituting the limits c and $c+\tau$ for 0 and τ. The work done in one period is therefore equal to the total energy in one wave-length.

Let us now consider a harmonic train advancing into still water. Take a plane on the wave-front. In one period the work done in

this plane, that is, the energy that crosses this plane, is sufficient to build up one wave-length. The head of the train advances therefore a distance λ in time τ and has the same velocity as the individual waves.

This is not generally true. If a stone is thrown into a pool and the group of waves that travels out from the point where it enters the water is watched, it will be noticed that the individual waves travel faster than the group. They grow up in the rear of the group, pass

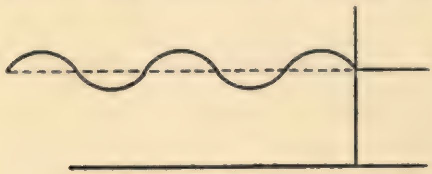

FIG. 55.

through it and in turn disappear in the still water in front of the group. The group velocity in this case is not so great as the wave velocity. But in the case of long waves the group velocity coincides with the wave velocity.

The group velocity gives the rate at which the energy is propagated.

It should be noted that the case represented in fig. 55 is an artificial one. A harmonic train could not be regular up to its very front.

§ 106. **Forced waves in a canal.**

Consider the equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2} + \mathsf{X}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

where X is a function of x and t, but not of y. Let X have the value $\mathsf{C}\sin(nt+mx)$ and assume $\xi = \mathsf{D}\sin(nt+mx)$. By substituting in the equation, we find that

$$\mathsf{D}(n^2 - v^2m^2) = -\mathsf{C},$$

and consequently that

$$\xi = -\frac{\mathsf{C}}{n^2 - v^2m^2}\sin(nt+mx).$$

To the above expression for ξ there can be added any solution of the equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2},$$

and equation (12) will still be satisfied. This additional part of the solution is called the free wave and the former part the forced wave. The complete solution is thus the sum of the forced and free waves.

When n^2 approaches v^2m^2, *i.e.* when the velocity of the impressed force approaches that of the free wave, the amplitude of the forced wave becomes very great. It does not, however, become infinite as the formula states, being prevented by viscosity, which is not considered by our elementary theory.

If $n^2 < v^2m^2$ the forced wave is direct, that is, it has a maximum when the impressed force has a maximum; if $n^2 > v^2m^2$ the forced wave is inverted and has a minimum when the impressed force has a maximum.

Equation (12) is of importance in the theory of the tides. For let there be a uniform canal round the earth's equator, and suppose for the sake of simplicity that the earth's axis is perpendicular to the ecliptic and that the moon's orbit is in the ecliptic. Let O be a fixed point on the earth's equator, let the distance of P measured round the equator from O be x, let n' be the angular velocity of the earth relative to the direction of the moon and let A be the point on the equator directly under the moon. Then, if t be measured from the instant when O was at A, the arc $\mathrm{AO} = an't$ and the arc $\mathrm{AP} = an't + x$, a being the radius of the earth.

It is only the component of the moon's tide-producing force tangential to the equator that has any effect in producing waves in the canal. We know from the equilibrium theory of the tides that the tide-producing force is a maximum and vertical at A and B, and that it is zero at J and K. The numerical value of the tangential component must therefore have a maximum value at intermediate points, G, H, F and E; at G and E it is towards the moon and at H and F it is away from it. The tangential component thus runs through all its values twice as we go once round the equator, and is hence proportional to $\sin 2\left(n't + \frac{x}{a}\right)$. The impressed force on the canal is therefore of the type considered in the earlier part of the present section.

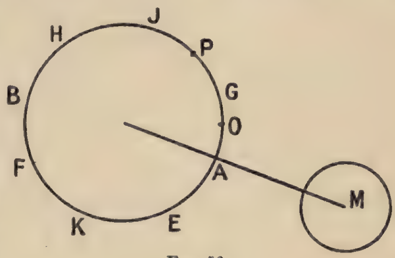

FIG. 56.

§ 107. **Gravity waves. General case.**

We shall now let fall the restriction that the waves are long and consider the general case of waves caused by gravity on the surface of a uniform canal of depth h. As in § 100, the origin will be taken in the bottom of the canal, Oy will be taken vertically upwards, Ox along the bottom, and ρ, the density of the liquid, will be taken constant.

We shall assume that the motion is irrotational. It will be remembered that in this case (cf. Chap. II.) the velocity is derived from a potential ϕ. In the problem under consideration, since it is one of two-dimensional motion, ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \quad \text{.............................(13)}$$

At the bottom we have the boundary condition

$$\frac{\partial\phi}{\partial y}=0 \text{ for } y=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

and at the surface we have another boundary condition which will be derived further down.

When ϕ is known, p, the pressure at any point (cf. § 41), is given by

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-gy-\tfrac{1}{2}q^2+\mathsf{F}(t).$$

This equation can be simplified for our purpose here. First of all the term $\frac{1}{2}q^2$ may be omitted, because in problems of wave motion the velocities are always supposed to be so small that their second powers may be neglected. Then the term $\mathsf{F}(t)$ may be supposed included in $\frac{\partial\phi}{\partial t}$. The equation thus becomes

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-gy.$$

In this equation ρ and g are constants, t and y being the usual independent variables. We can, however, take t and p as independent variables. Then differentiation with respect to t gives

$$\underset{p\text{ const.}}{\frac{\partial}{\partial t}}\left(\underset{y\text{ const.}}{\frac{\partial\phi}{\partial t}}\right)-g\frac{\partial y}{\partial t}=0.$$

This equation holds for any surface for which p is constant, and hence for the air-liquid surface. Now $\frac{\partial y}{\partial t}$ is equal to the vertical component of the velocity of this surface and can be put equal to $-\frac{\partial\phi}{\partial y}$, and $\frac{\partial}{\partial t}_{y\text{ const.}}$ may be written for $\frac{\partial}{\partial t}_{p\text{ const.}}$, since the velocity is supposed to be small. We thus have the boundary condition at the upper surface of the liquid,

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial y}=0 \text{ for } y=h. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Assume $$\phi=\mathsf{F}(y)\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right),$$

$\mathsf{F}(y)$ being independent of x and t, and substitute in (13). This gives

$$\frac{\partial^2\mathsf{F}}{\partial y^2}-\left(\frac{2\pi}{\lambda}\right)^2\mathsf{F}=0,$$

the solution of which is

$$\mathsf{F}(y)=\mathsf{P}\cosh\frac{2\pi y}{\lambda}+\mathsf{Q}\sinh\frac{2\pi y}{\lambda}.$$

Hence $$\phi=\left(\mathsf{P}\cosh\frac{2\pi y}{\lambda}+\mathsf{Q}\sinh\frac{2\pi y}{\lambda}\right)\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right).$$

From (14) we find that Q must equal zero, and from (15) that

$$\left(\frac{2\pi}{\tau}\right)^2 \cosh\frac{2\pi h}{\lambda} = g\frac{2\pi}{\lambda}\sinh\frac{2\pi h}{\lambda} \quad \text{or} \quad v^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}.$$

There are two special cases of this formula. If the waves are long, h/λ is small, $2\pi h/\lambda$ can be written for $\tan 2\pi h/\lambda$ and $v^2 = gh$, a result we have obtained already. If h/λ is large, the tangent may be put $=1$. The velocity is then given by $v^2 = \frac{g\lambda}{2\pi}$. This is the case of "deep-sea" waves.

Let ξ, η denote the displacement of the particle originally at x, y.

Then
$$\frac{\partial \xi}{\partial t} = -\frac{\partial \phi}{\partial x} = -\mathsf{P}\frac{2\pi}{\lambda}\cosh\frac{2\pi y}{\lambda}\sin\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right),$$
$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} = -\mathsf{P}\frac{2\pi}{\lambda}\sinh\frac{2\pi y}{\lambda}\cos\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right).$$

These equations give

$$\xi = +\frac{\mathsf{P}}{v}\cosh\frac{2\pi y}{\lambda}\cos\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right), \quad \eta = -\frac{\mathsf{P}}{v}\sinh\frac{2\pi y}{\lambda}\sin\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right),$$

whence
$$\frac{\xi^2}{\cosh^2\frac{2\pi y}{\lambda}} + \frac{\eta^2}{\sinh^2\frac{2\pi y}{\lambda}} = \frac{\mathsf{P}^2}{v^2}.$$

The path of the particle is thus an ellipse, the longer axis being horizontal. In the case of deep-sea waves it becomes a circle for particles on the surface.

§ 108. **Two horizontal dimensions. Stationary waves in a rectangular vessel.**

Let a be the length of the vessel, b its breadth and h the depth of the liquid. Then ϕ must satisfy the following conditions:

(1) $\nabla^2\phi = 0$;

(2) $\frac{\partial \phi}{\partial x} = 0$ for $x = 0$, $x = a$;

(3) $\frac{\partial \phi}{\partial z} = 0$ for $z = 0$, $z = b$;

(4) $\frac{\partial \phi}{\partial y} = 0$ for $y = 0$;

(5) $\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0$ for $y = h$.

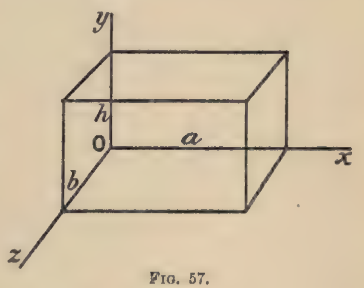

FIG. 57.

Assume
$$\phi = \begin{matrix}\cosh\\ \sinh\end{matrix}\,\gamma y \begin{matrix}\cos\\ \sin\end{matrix}\,\alpha x \begin{matrix}\cos\\ \sin\end{matrix}\,\beta z \begin{matrix}\cos\\ \sin\end{matrix}\,\gamma v t.$$

Then, from (1), $\alpha^2 + \beta^2 - \gamma^2 = 0$, hence $\gamma = \sqrt{\alpha^2 + \beta^2}$. To satisfy (4) we must choose the cosh. To satisfy (2) and (3) we must take $\cos\alpha x \cos\beta z$

and write $\alpha = m\pi/a$, $\beta = n\pi/b$, where m and n are integers. The typical solution is then

$$\phi = \cosh \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\, \pi y \cos \frac{m\pi x}{a} \cos \frac{n\pi z}{b} \begin{matrix}\cos \\ \sin\end{matrix} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\, \pi v t,$$

v being given from (5) by

$$v^2 = \frac{g}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\, \pi} \tanh \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\, \pi h.$$

It is obvious that different terms of the above type can be combined so as to give initially any prescribed value to the elevation on the surface and to the vertical component of the velocity on the surface.

EXAMPLES.

1. Show that in the case of deep-sea waves each particle of the liquid describes a circle, and determine the relation of the radius of the circle to the depth below the surface of the liquid.

2. A straight canal of depth h and length l has a rectangular cross-section, and its ends are vertical and at right angles to its length. Show that the periods of the longitudinal waves that can be propagated in it are obtained by giving positive integral values to n in the expression

$$2\sqrt{\left(\frac{\pi l}{ng} \coth \frac{n\pi h}{l}\right)}.$$

3. Find a solution of the differential equation for long waves in a canal of length l closed at one end and communicating at the other with a tidal sea, the level of which varies according to the equation

$$\eta = a \cos(nt + \theta).$$

4. The space between two infinite horizontal planes is filled with two fluids, one of density ρ and depth h, and the other of density ρ' and depth h'. Prove that the velocity of a long wave on the surface of separation is

$$\sqrt{\left\{\frac{g(\rho - \rho')hh'}{h'\rho + h\rho'}\right\}}.$$

5. Waves are propagated in a canal of depth h. What relation must exist between h and λ in order that (1) the formula for long waves, (2) the formula for deep-sea waves should represent the velocity correctly to 1 per cent.?

6. Discuss the characteristics of the motion for which (cf. § 47)

$$\phi + i\psi = Ae^{im(x+iy)} \sin nt.$$

7. The section of a canal is semi-circular, of radius a. It is full to the horizontal diameter, and above that the banks are vertical. Prove that the velocity of propagation of long waves in it is $\frac{1}{2}(\pi g a)^{\frac{1}{2}}$.

8. A long wave in a liquid of depth h represented by

$$y=f(vt-x)$$

is reflected by a vertical wall at right angles to its direction of propagation. Find the thrust, if any, exerted on the wall.

9. Use the formula (cf. § 45)

$$2\,\mathrm{T}=\rho\int\phi\,\frac{\partial\phi}{\partial n}\,ds$$

to investigate the kinetic energy of the motion given by

$$\phi=\mathrm{P}\cosh\frac{2\pi y}{\lambda}\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right),$$

the origin being taken at the bottom, and the surface being given by $y=h$.

10. Prove that the group velocity of deep-sea waves is half their wave velocity.

§ 109. **Sound waves in a gas.**

We found in § 33 that the equation of continuity for a fluid, when expressed in its most general form, was

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0.$$

Also the equations of motion were

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\mathrm{X}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

with two similar equations. We shall now apply these equations to the case of wave motion in a perfect gas, and shall make the following assumptions:

(1) The motion is irrotational, *i.e.* $u=-\dfrac{\partial\phi}{\partial x}$, $v=-\dfrac{\partial\phi}{\partial y}$ and $w=-\dfrac{\partial\phi}{\partial z}$.

(2) The velocities are so small that their squares and products can be neglected.

(3) The "body" forces, X, Y, Z, can be neglected.

For ρ write $\rho_0(1+s)$, where ρ_0 is the initial value of the density; s is called the condensation. As the alterations of density in a sound wave are slight, s is small, and we can neglect the $\dfrac{\partial s}{\partial x}$, $\dfrac{\partial s}{\partial y}$, $\dfrac{\partial s}{\partial z}$ terms in the continuity equation. It then becomes

$$\frac{\partial s}{\partial t}=(1+s)\nabla^2\phi \quad\text{or}\quad \frac{\partial s}{\partial t}=\nabla^2\phi, \quad\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

since s is small. To the same order of approximation the first equation of motion becomes

$$\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x}=\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

If the two similar equations be written down and the three equations be multiplied respectively by dx, dy, dz and added, we obtain

$$\frac{\partial}{\partial t}d\phi=\frac{dp}{\rho} \quad \text{or} \quad \frac{\partial\phi}{\partial t}=\int\frac{dp}{\rho}+\mathrm{C}. \qquad (17)$$

Let us now assume that Boyle's law holds and that $p=c\rho$. Then $dp=c\rho_0\,ds$, consequently

$$\int\frac{dp}{\rho}=\int\frac{c\,ds}{1+s}=c\log(1+s),$$

and, from (17),

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial}{\partial t}\int\frac{dp}{\rho}=\frac{c}{1+s}\frac{\partial s}{\partial t}.$$

This may be written

$$\frac{\partial^2\phi}{\partial t^2}=c\frac{\partial s}{\partial t}, \qquad (18)$$

since s is small. Combining (16) and (18), we obtain

$$\frac{\partial^2\phi}{\partial t^2}=c\nabla^2\phi, \qquad (19)$$

This is the general equation for the propagation of wave motion.

Assume

$$\phi=\mathrm{A}\sin\frac{2\pi}{\tau}\left(t-\frac{lx+my+nz}{v}\right).$$

It is obvious that this represents a plane wave propagated with velocity v in the l, m, n direction, for the surfaces of equal phase are given by

$$lx+my+nz=vt.$$

Substitute this value of ϕ in equation (19), and we find, since

$$l^2+m^2+n^2=1, \quad \text{that} \quad v^2=c.$$

But $c=p/\rho$. Hence $v=\sqrt{p/\rho}$, and the velocity of the wave can be calculated.

This is the well-known expression derived by Newton for the velocity of sound in a gas. It did not agree with experiment, the result given by it being too small. The correct expression was first derived by Laplace, who showed, that in the case of sound waves the condensation and rarefaction take place so rapidly, that the heat produced has not time to disappear by conduction. The temperature of each element of mass will thus not be constant, but the quantity of heat contained in it will. The change is not an isothermal but an adiabatic one. In this case, as is shown in Chapter VI., the relation existing between the pressure and volume is not $p=c\rho$, but $p=c\rho^\kappa$, c and κ being constants; κ is the ratio of the specific heat at constant pressure to the specific heat at constant volume and for air, oxygen, hydrogen and nitrogen it has the value 1·41.

If $$p=c\rho^\kappa,\quad dp=c\rho_0{}^\kappa\kappa(1+s)^{\kappa-1}ds$$

and $$\int\frac{dp}{\rho}=c\kappa\rho_0{}^{\kappa-1}\int(1+s)^{\kappa-2}ds=\frac{c\kappa\rho_0{}^{\kappa-1}}{\kappa-1}(1+s)^{\kappa-1}.$$

Consequently $$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial}{\partial t}\int\frac{dp}{\rho}=c\kappa\rho_0{}^{\kappa-1}(1+s)^{\kappa-2}\frac{\partial s}{\partial t}.$$

Now $(1+s)^{\kappa-2}$ may be put $=1$ and $c\rho_0{}^{\kappa-1}=p_0/\rho_0$. Hence

$$\frac{\partial^2\phi}{\partial t^2}=\kappa\frac{p_0}{\rho_0}\frac{\partial s}{\partial t}=\frac{\kappa p_0}{\rho_0}\nabla^2\phi,$$

and the velocity of the wave is $\sqrt{\kappa p_0/\rho_0}$. This result agrees well with experiment.

Consider, again, the expression for the velocity potential in the case of the plane wave travelling in the l, m, n direction, namely,

$$\phi=\mathsf{A}\sin\frac{2\pi}{\tau}\left(t-\frac{lx+my+nz}{v}\right).$$

It gives $$u=-\frac{\partial\phi}{\partial x}=\mathsf{A}\frac{2\pi l}{\lambda}\cos\frac{2\pi}{\tau}\left(t-\frac{lx+my+nz}{v}\right)$$

with similar expressions for v and w, whence we derive the result that

$$\frac{u}{l}=\frac{v}{m}=\frac{w}{n},$$

that the velocity is perpendicular to the wave front. The wave is thus longitudinal.

§ 110. **Transverse waves.**

Consider the expressions

$$\eta=b\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right),\quad \zeta=c\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)+\delta\right\}. \quad \ldots\ldots\ldots\ldots(20)$$

They represent two waves being propagated in the positive x direction. If η and ζ denote respectively the displacements parallel to $\mathsf{O}y$ and $\mathsf{O}z$ of the particle at x, y, z, the resultant displacement is transverse to the direction of propagation, and both waves together are said to constitute an elliptically polarised wave, because, as the wave passes, the particles describe ellipses parallel to the yz-plane.

To find the equation to these ellipses write

$$\zeta=c\left\{\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\cos\delta+\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\sin\delta\right\},$$

i.e. $$\frac{\zeta}{c}=\frac{\eta}{b}\cos\delta+\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\sin\delta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

and eliminate $\left(t-\frac{x}{v}\right)$ between (20) and (21). This gives

$$\left(\frac{\eta}{b}\right)^2+\left(\frac{\zeta}{c\sin\delta}-\frac{\eta}{b}\cot\delta\right)^2=1 \quad \text{or} \quad \frac{\eta^2}{b^2\sin^2\delta}-\frac{2\eta\zeta\cos\delta}{bc\sin^2\delta}+\frac{\zeta^2}{c^2\sin^2\delta}=1,$$

which is of course an ellipse since the asymptotes are imaginary.

The y and z-axes can be chosen so as to make the product term disappear. Then we have $\cos\delta=0$, and the elliptically polarised wave referred to the new axes can be written in the form

$$\eta=b\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right), \quad \zeta=c\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\pm\frac{\pi}{2}\right\}.$$

An elliptically polarised wave is the most general type of transverse wave. When $b=c$, the orbit of each particle is a circle and the wave is said to be circularly polarised. When the phase difference is zero, the orbit becomes a straight line and the wave is said to be plane polarised.

EXAMPLES.

1. A vibration of frequency n is rendered intermittent in frequency m by the interposition of an obstacle, so that it can be represented by the expression

$$(1+\cos 2\pi mt)\cos 2\pi nt.$$

Show that the intermittent vibration is equivalent to three simple vibrations of definite frequencies, which find.

Apply this to the explanation of the two sounds, one above, the other below the pitch of a fork, which are produced when the sound of the latter is intercepted by a perforated revolving screen.

2. A sound wave is given by

$$\phi=\mathsf{A}\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right).$$

If p_0, $p_0+\delta p$ denote the pressures when the air is at rest and in motion respectively, the rate at which energy is transmitted across unit area in the wave front is $-(p_0+\delta p)\frac{\partial\phi}{\partial x}$. Show that $\delta p=\rho_0\frac{\partial\phi}{\partial t}$ and that consequently the average rate of flow of energy across unit area in the wave front is

$$\frac{2\pi^2\mathsf{A}^2\rho_0}{v\tau^2}.$$

3. Find an expression for the velocity potential for stationary waves in a cylindrical pipe, the wave fronts being perpendicular to the sides of the cylinder. Assume (1) that the pipe is closed at both ends, (2) that it is open at one end and closed at the other. (Boundary condition at a closed end $\frac{\partial\phi}{\partial x}=0$; at an open end $\phi=0$.)

4. Show that
$$\phi=\frac{1}{r}f(vt-r),$$
and any differential coefficient of it with respect to x, y, z are solutions of the equation
$$\frac{\partial^2\phi}{\partial t^2}=v^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right).$$

5. Transform the equation of wave propagation
$$\frac{\partial^2\phi}{\partial t^2}=v^2\nabla^2\phi$$
to the proper form for symmetry about an axis, namely
$$\frac{\partial^2(r\phi)}{\partial t^2}=v^2\frac{\partial^2(r\phi)}{\partial r^2}$$

Adopting the particular solution
$$r\phi=\sin\frac{2\pi}{\tau}\left(t-\frac{r}{v}\right),$$
show that only at a distance from the source equal to many wave-lengths is $\frac{\partial\phi}{\partial r}$ inversely proportional to the distance, while near the source a different law holds good.

6. A gas is enclosed within a rigid spherical envelope of radius a and vibrates symmetrically about the centre. Show that the frequency is given by $\frac{nv}{2\pi}$, where n is given by $\tan na=na.$

CHAPTER V.

ELECTROMAGNETIC THEORY.

§ 111. It is shown in the elementary text-books, that the attraction between permanent bar magnets may be explained by supposing charges of positive and negative magnetism to reside at the ends of each magnet and by supposing that like charges repel and unlike charges attract one another. In the case of a thick magnet these charges occupy regions near the ends; these regions are the parts that the lines of force emanate from, when the field of the magnet is plotted with a compass needle, and they are called the poles of the magnet. If the magnets are long, thin and uniformly magnetized, the poles contract to points exactly at the ends, and we base our definition of unit quantity of magnetism or pole strength on this case. Two like poles of equal strength are said to have unit quantity of magnetism, when they repel one another at a distance of one centimetre with a force of one dyne, both being in air.

It has been proved by Coulomb with the torsion balance and also by Gauss by measuring the attraction between two magnets in the "A" and "B" tangential positions, that the force between two poles varies inversely as the square of the distance between them. It is thus analogous to gravitational attraction. In order to define the field strength or the magnetic intensity (H) at a point in the field of a magnet or system of magnets, we suppose a positive pole of strength m placed at that point; then $\mathsf{H}m$ gives the force with which the field acts on the pole. It should be noted that $\mathsf{H}m$ has the dimensions of force; H has not. The potential at a point in the field (V) is the work that would have to be done against the forces of the field in bringing unit positive pole from infinity to that point. Thus, in analogy with gravitational attraction,

$$\mathsf{H} = -\frac{\partial \mathsf{V}}{\partial s}.$$

The magnetic moment of a bar magnet (M) is equal to the pole-strength multiplied by the distance between the poles. If the poles do not occupy points but cover a definite region at each end of the magnet, the moment is obtained by dividing the total quantity of magnetism into elements and by multiplying each typical element dm by l, the distance between it and the corresponding element at the

other end of the magnet. Then $\mathsf{M} = \Sigma l\,dm$. The intensity of magnetization of a magnet (**I**) is defined as its magnetic moment per unit volume. If the magnet is a thin cylindrical one of length l, and cross-sectional area a, the volume is la and the intensity of magnetization is given by

$$\mathbf{I} = \frac{\mathsf{M}}{la} = \frac{ml}{la} = \frac{m}{a},$$

that is, it is the surface density of magnetism on either end of the magnet.

§ 112. **Magnetic potential due to a small magnet.**

Let A and B represent the poles of a small magnet, m the charge of magnetism at A and $-m$ that at B. Let C be the middle point of the magnet and let $l = \mathsf{AB}$. Consider the potential at a point P, at such a distance from the magnet that PC is large in comparison with AB. The potential at P is made up of two parts; it is the sum of the potentials due to the positive pole and the negative pole, that is, it is equal to

$$\frac{m}{\mathsf{AP}} - \frac{m}{\mathsf{BP}}.$$

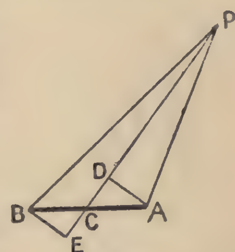

Fig. 58.

Draw AD and BE perpendicular to CP, write r for CP and let angle PCA be θ. Then, since r is large in comparison with l, triangle DPA may be regarded as isosceles. Hence

$$\mathsf{AP} = \mathsf{DP} = \mathsf{CP} - \mathsf{CD} = \mathsf{CP} - \mathsf{CA}\cos\theta = r - \frac{l}{2}\cos\theta,$$

and similarly $\mathsf{BP} = r + \dfrac{l}{2}\cos\theta$. Therefore the potential at P,

$$\frac{m}{\mathsf{AP}} - \frac{m}{\mathsf{BP}} = m\left(\frac{1}{r - \frac{l}{2}\cos\theta} - \frac{1}{r + \frac{l}{2}\cos\theta}\right) = \frac{ml\cos\theta}{r^2 - \frac{l^2\cos^2\theta}{4}} = \frac{\mathsf{M}\cos\theta}{r^2},$$

since $\dfrac{l^2\cos^2\theta}{4}$ may be neglected in comparison with r^2. It should be noted that θ is the angle which the direction of the point makes with the positive direction of the axis of the magnet, that is, the line drawn from its negative to its positive pole.

§ 113. **Magnetic shell. Magnetic potential due to a uniform shell.**

A magnetic shell is a very thin sheet of magnetizable substance, magnetized at each point in the direction of the normal to the sheet at that point.

The strength of the shell (ϕ) at any point is the product of the intensity of magnetization at that point into the thickness of the shell measured along the normal.

If t denotes the thickness of the shell, $\phi = \mathrm{I}t$. Hence ϕ at any point is the magnetic moment of unit area of the shell at that point.

The shell is said to be uniform when ϕ is constant all over it.

Suppose that it is required to determine the potential due to the shell at a point P outside it, the distance of P from the shell being large in comparison with the thickness of the latter. Consider an element of surface at the point A, of area a, and by drawing lines normal to the surface at every point on the boundary of this area, cut a small magnet out of the shell. BA is the axis of this magnet. Its pole strength is $\mathrm{I}a$ and its length is t. Consequently its moment is $\mathrm{I}at$ or ϕa. If AP be denoted by r, the potential at P due to this elementary magnet is

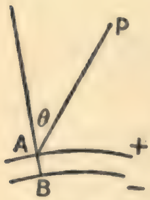

Fig. 59.

$$\frac{\phi a \cos\theta}{r^2},$$

where θ has the value shown in the diagram. Now $a\cos\theta$ is the projection of the area of the end of the magnet on a plane at right angles to AP, and thus $\frac{a\cos\theta}{r^2}$ is the solid angle subtended at P by this area. Let this solid angle be denoted by $d\Omega$. Then the potential due to the elementary magnet is

$$\phi\, d\Omega.$$

If we suppose now that the shell is divided into a number of such elementary magnets, it is clear that the potential due to the whole shell will be

$$\phi\Omega,$$

where Ω is the solid angle subtended at P by the whole shell. Ω depends only on the shape of the boundary of the shell. From the method of establishing the result it is evident that the potential is positive on the positive side of the shell and negative on the other side.

Let ACB be a section of a uniform shell. Let P and Q be two points close up to the shell on opposite sides of it, P being on the positive side. Let us suppose that it is required to determine the difference of potential between P and Q.

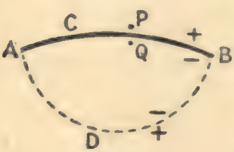

Fig. 60.

Let $\mathrm{V_P}$ be the potential at P and $\mathrm{V_Q}$ the potential at Q. Suppose now that another shell of the same strength and with the same boundary is placed in the position ADB indicated by the dotted line. The negative surfaces of the two shells face one another and together they may be regarded as constituting one closed shell.

Since P and Q are close together the potential produced at P and Q by the part ADB is approximately the same. Denote it by V. Then the resultant potential at P is $\mathrm{V_P} + \mathrm{V}$ and the resultant potential at Q is $\mathrm{V_Q} + \mathrm{V}$. But the solid angle subtended at P by the whole

closed shell is zero, for the one part annuls the other. And the solid angle subtended at Q is -4π. Hence

$$V_P + V = 0, \quad V_Q + V = -4\pi\phi,$$

whence

$$V_P - V_Q = 4\pi\phi.$$

§ 114. **Ampère's theorem.**

In 1820 Oersted discovered that a current-carrying circuit produced a magnetic field, and in 1823 Ampère enunciated the law that gives the magnetic intensity at any point in the field of such a circuit. It runs as follows: "Every linear conductor carrying a current is equivalent to a simple magnetic shell, the bounding edge of which coincides with the conductor and the moment of which per unit of area, *i.e.* the strength of the shell, is proportional to the strength of the current." The direction of magnetization of the shell is related to the direction of the current in such a way, that if an observer stands on the positive side of the shell near the edge facing the direction in which the current is flowing, the area of the shell is on his left hand. The best proof of Ampère's theorem lies in the fact that it is the basis of the whole science of electromagnetism. Its results are thus being compared daily with experience, and no case has been discovered in which it does not hold.

The electromagnetic unit of current is defined so that when the current is expressed in it, it is numerically equal to the strength of the equivalent shell. In other words, the constant of proportionality becomes unity. We shall, however, use electrostatic units. The theorem can then be written

$$\phi = i/c,$$

ϕ being the strength of the equivalent shell, c being a constant and i the strength of the current in electrostatic units.

The magnetic intensity in the field of an electric current depends only on the strength of the current and not on the nature of the medium filling the field. The magnetic intensity in the field of a magnet depends on the nature of the medium in the field. Hence Ampère's theorem is intended to hold only for a current-carrying circuit situated in air.

Ampère's theorem, of course, does not hold for points inside the shell.

§ 115. **Work done in carrying unit positive pole round closed path in field of current.**

Let A be the trace of the wire carrying the current and let AB be a section of the equivalent magnetic shell. Suppose that the positive pole is carried round the circuit R. Then the work done is zero, because the circuit is analogous to an external circuit in the field of a mass of gravitating matter. Suppose now that the pole goes from P to Q. The difference of potential between P and Q is $4\pi\phi = 4\pi i/c$. If we were to bring the pole from Q to P through the shell, this difference of potential would be lost and the resultant work done in the circuit would be zero. But it must be remembered, Ampère's

theorem does not hold for points inside the shell. Consequently, when the pole arrives at Q, assume the shell removed and let the pole continue its path from Q to P in air. The length QP is so short, that the work done on it can be neglected, and thus the whole work done in the circuit is $4\pi i/c$. If the path of the pole is a closed one threading the wire carrying the current n times, the work done on the

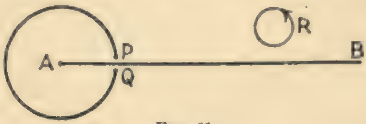

FIG. 61.

pole is $4\pi ni/c$. If there are permanent magnets in the field, the result still holds, for they cannot influence the work done on a closed path.

The theorem proved above is called the "first circuital theorem."

We have noticed incidentally, that the magnetic potential of a current-carrying circuit is multiple-valued, while that of the equivalent shell is single-valued.

§ 116. **Case of a right circular cylindrical conductor.**

Suppose that we have a homogeneous, right circular cylindrical conductor, of radius a, infinitely long, with a steady current flowing in it, and let the direction of the conductor be perpendicular to the plane of the paper. There is no magnetizable matter in the field. It is required to find H, the magnetic intensity, both inside and outside the conductor.

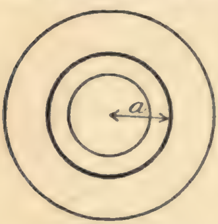

FIG. 62.

The direction of the magnetic intensity will be in the plane of the paper and everywhere tangential to circles with their centres in the axis of the cylinder. Take therefore a circular path of radius r, r being greater than a. The work done on the unit positive pole in taking it round this path is $2\pi r\mathsf{H}$. By the first circuital theorem this is equal to $4\pi i/c$. Hence

$$\mathsf{H} = \frac{2i}{cr}.$$

This gives the value of H in the air outside the conductor.

Suppose now that r is less than a. Since the current is uniformly distributed over the conductor only a fraction $(r/a)^2$ will flow then through the circuit. The work done in taking unit positive pole round the circuit is therefore $\dfrac{4\pi ir^2}{ca^2}$, and H is given by

$$\mathsf{H} = \frac{2ir}{ca^2}.$$

This gives the value inside the conductor. Both values of course coincide for $r = a$.

§ 117. **First circuital theorem. More general form.**

Let all space be filled with a conducting medium not necessarily homogeneous and let there be electric currents everywhere. At the point x, y, z let the components of H be α, β, γ, and let the components of current per unit area be u, v, w. That is, if we set up an area of 1 sq. cm. at right angles to the x-axis, u gives the quantity of electricity measured in electrostatic units which flows through it in one second.

Draw any closed circuit in this medium. Then, by Stokes' theorem,

$$\int\{\alpha\, dx+\beta dy+\gamma dz\}=\iint\left\{l\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\right)+m\left(\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x}\right)+n\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)\right\}d\mathsf{S}.$$

The expression on the left is the line integral of the tangential component of H taken round this circuit; the expression on the right is the surface integral of the normal component of the curl of H taken over any surface bounded by the circuit. But, by the first circuital theorem,

$$\int\{\alpha\, dx+\beta dy+\gamma dz\}=\frac{4\pi}{c}\iint\{lu+mv+nw\}\,d\mathsf{S}.$$

The surface integral on the right gives the total current through the circuit. Combining this equation with the previous one, we obtain

$$\frac{4\pi}{c}\iint\{lu+mv+nw\}\,d\mathsf{S}=\iint\left\{l\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\right)+m\left(\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x}\right)+n\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)\right\}d\mathsf{S}.$$

The above equation is true, no matter what the boundaries and shape of the surface are. It holds true for every element of it, no matter what values l, m, n may have; we may therefore equate the two integrands. Thus the equation decomposes into the following three:

$$\frac{4\pi u}{c}=\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\quad \frac{4\pi v}{c}=\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\quad \frac{4\pi w}{c}=\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y},$$

which hold for every point in the medium. They are equations which enable us to determine the current when the magnetic intensity is known.

§ 118. **The displacement current.**

If we differentiate the first of the above three equations with respect to x, the second with respect to y, the third with respect to z, and add, the right-hand side vanishes, and the left-hand becomes

$$\frac{4\pi}{c}\frac{\partial u}{\partial x}+\frac{4\pi}{c}\frac{\partial v}{\partial y}+\frac{4\pi}{c}\frac{\partial w}{\partial z}=0\quad\text{or}\quad\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

if the common factor is cancelled out.

This equation states that the divergence of the electric current is equal to zero. In hydrodynamics there can be no sources and no sinks where the divergence of the velocity is equal to zero. The stream

lines can have no ends; they must all be closed curves. According to the above equation it is the same with electric currents. They must all be closed. They can start and end nowhere.

Up to the present we have tacitly assumed the current to be a conduction current, a procession of electric charges along a wire. If a battery is connected with a resistance box and a steady current is sent through the latter, then this conduction current is a closed one. But if an insulated, uncharged piece of wire is suddenly charged by connecting one end of it to one side of a battery, then the conduction current is not a closed one. For, isolate any small portion of the wire by drawing a sphere round it, and consider any small interval dt during the time that the wire is charging. As the wire is filling up with electricity, the quantity of electricity on the part inside the sphere will increase by a definite quantity $d\mathsf{Q}$ during this time. Let σ be the area of cross-section of the wire, u the current per unit area of cross-section and dx the length of the element of wire. Then the rate at which electricity is flowing into the element is given by σu and the rate at which it is flowing out by $\sigma\left(u+\frac{\partial u}{\partial x}dx\right)$. The rate at which it is being gained is $-\sigma\frac{\partial u}{\partial x}dx$; this is equal to $\frac{\partial \mathsf{Q}}{\partial t}$. Thus for no point on the wire is $\frac{\partial u}{\partial x}=0$.

The equation at the beginning of this section is therefore not true when applied to varying conduction currents. The main feature of Clerk Maxwell's theory of electricity is that the conception of electric current is extended so as to make the equation universally true.

The unit quantity of electricity on the electrostatic system is that charge which repels an equal and like charge at a distance of one centimetre with a force of one dyne, both being in air. The force between two electrostatic charges varies inversely as the square of the distance between them. In order to define E, the electric intensity or the field strength at a point in the field of a system of charged conductors, we suppose a small positive charge e placed at that point without disturbing the distribution of the charges already in the field; then $\mathsf{E}e$ gives the force with which the field acts on the charge. It should be noted that $\mathsf{E}e$ has the dimensions of force; E has not. The potential at a point in the field (V) is the work that would have to be done against the forces of the field in bringing unit positive charge from infinity to that point. As formerly,

$$\mathsf{E}=-\frac{\partial \mathsf{V}}{\partial s}.$$

The capacity of a condenser varies with the specific inductive capacity (k) of the medium between the plates. We define k by taking it proportional to the capacity of the condenser and making the value for air unity. Since the capacity varies as k, the difference of potential between the two plates of the condenser varies inversely

as k. Consequently the electric intensity in the medium between the plates also varies inversely as k.

We are thus led to the conclusion, that if we have a point charge e situated in a medium of specific inductive capacity k, the electric intensity E at a point P distant r from it is given by

$$\mathsf{E}=\frac{e}{kr^2}.$$

It is now necessary to introduce a new vector, the electric displacement (D) at P, which is defined by

$$\mathsf{D}=\frac{k}{4\pi}\mathsf{E}.$$

In the case of the above point charge,

$$\mathsf{D}=\frac{e}{4\pi r^2}.$$

If the medium is isotropic, as we have tacitly assumed, D has everywhere the same direction as E. D is independent of the medium in which the point charge happens to be placed. It is supposed to measure a state of strain at the point. The energy of the point charge is stored up in its field, and a state of strain is set up everywhere in the field. When this state of strain is set up, something is displaced at the point; hence the name.

The rate at which the displacement through an area is increasing gives the displacement current in the direction perpendicular to that area. The displacement current was introduced by Clerk Maxwell. The true current is the sum of the conduction and displacement currents and the latter is to be regarded as producing a magnetic field in the same way as the former. For example, a point charge is moving with velocity v in a straight line. About this straight line as axis a circle is described. If the velocity of the point charge is small in comparison with the velocity of light, the displacement through the circle is proportional to the solid angle subtended at the point charge by the circle. As the charge approaches the circle, this angle increases; the displacement through the circle increases, and there is consequently a displacement current through the circle, which produces a magnetic intensity tangential to the latter.

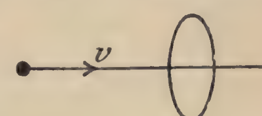

FIG. 63.

If we return now to the case we were considering earlier in the section, the charge inside the small sphere has increased by $d\mathsf{Q}$ in dt, *i.e.* the surface integral of displacement current over the sphere is $\frac{\partial \mathsf{Q}}{\partial t}$ outwards. This expression also gave the resultant conduction current into the sphere. Hence the divergence of the true current is zero.

Let κ denote the conductivity of the medium; then $\kappa\mathsf{E}$ denotes the resultant conduction current per unit area. E is not the electromotive force or the difference of potential between the ends of a wire, but the space rate of change of the latter or the potential gradient. κ gives the quantity of electricity flowing per second through an area of one square centimetre when the potential gradient at right angles to that area is unity. Let the components of E be X, Y, Z; then the components of conduction current are $\kappa\mathsf{X}$, $\kappa\mathsf{Y}$, $\kappa\mathsf{Z}$. The components of displacement are $\frac{k}{4\pi}\mathsf{X}$, $\frac{k}{4\pi}\mathsf{Y}$, $\frac{k}{4\pi}\mathsf{Z}$, and the components of the displacement current

$$\frac{k}{4\pi}\frac{\partial \mathsf{X}}{\partial t},\quad \frac{k}{4\pi}\frac{\partial \mathsf{Y}}{\partial t},\quad \frac{k}{4\pi}\frac{\partial \mathsf{Z}}{\partial t}.$$

If we substitute the total current for u, v, w in the first circuital equation, we obtain

$$\frac{4\pi\kappa}{c}\mathsf{X}+\frac{k}{c}\frac{\partial \mathsf{X}}{\partial t}=\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\qquad \frac{4\pi\kappa}{c}\mathsf{Y}+\frac{k}{c}\frac{\partial \mathsf{Y}}{\partial t}=\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},$$

$$\frac{4\pi\kappa}{c}\mathsf{Z}+\frac{k}{c}\frac{\partial \mathsf{Z}}{\partial t}=\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y},$$

the first three equations for the electromagnetic field. They may be summarised as

$$\frac{4\pi\kappa}{c}\mathsf{E}+\frac{k}{c}\frac{\partial \mathsf{E}}{\partial t}=\text{curl}\,\mathsf{H}.$$

EXAMPLES.

1. Find an expression for the magnetic intensity due to a small magnet at any point, the distance of which is great in comparison with the size of the magnet. Show that for a given distance the maximum value of the intensity is twice its minimum.

2. If the magnetic intensity varies inversely as the n^{th} power of the distance, show that if we have two small magnets, the couple on the second when the first magnet is "end on" to it, is n times the couple when the first magnet is "broadside on," the distance between the magnets being the same in each case.

3. Two circles of wire, of radii a, b, are placed in parallel planes perpendicular to the line joining their centres which are at a distance x apart. Show that if γ is the current in each circle in electromagnetic units and b/a is small, the force exerted by either circle on the other is approximately

$$\frac{6\pi^2\gamma^2a^2b^2x}{(a^2+x^2)^{\frac{5}{2}}}.$$

4. An infinitely long right circular solenoid has n turns of wire wound round each unit of length. The current in the wire is γ electromagnetic units. Show that the magnetic intensity inside the solenoid is given by

$$\mathsf{H}=4\pi n\gamma.$$

5. Show that the potential energy of a uniform shell due to its introduction into a magnetic field is

$$-\phi \mathsf{N},$$

where ϕ is the strength of the shell and N is the number of lines of magnetic intensity due to the external system passing through the shell in the direction of its magnetization.

6. A straight wire extends to infinity from a point A, and carries a current γ (electromagnetic units). From A it is continued in the other direction to infinity by a plane sheet in the form of a uniform circular sector of angle 2θ, which is bisected by the prolongation of the direction of the wire. Prove that the magnetic field intensity at a point P on a line through A perpendicular to the plane of the sector and distant a from it is

$$\frac{\gamma}{a}\left(1+\frac{\sin\theta}{\theta}\right).$$

7. An insulated straight wire is embedded in an infinite conducting medium and a current γ (electromagnetic units) flows in it. Show that the magnetic intensity at any point P is given by

$$\frac{\gamma(\cos\theta_1-\cos\theta_2)}{h},$$

where h is the perpendicular distance from P to the wire and θ_1, θ_2 are the angles which the wire makes with AP, BP the lines joining its ends to P.

8. A point charge of electricity is situated on the axis of a circle of radius c at a distance a from the plane of the circle. Show that the total displacement through the circle is

$$\frac{e}{2}\left(1-\frac{a}{(a^2+c^2)^{\frac{1}{2}}}\right).$$

Hence find the time rate of change of displacement through the circle if the charge e is travelling with velocity v along the axis, and the corresponding line integral of magnetic intensity round the circle.

§ 119. Current induction.

If a coil of wire is connected in circuit with a galvanometer and the pole of a magnet is thrust into the coil, a current is set up through the galvanometer. This current endures as long as the magnet is moving and ceases whenever the magnet comes to rest. If again, instead of thrusting the magnet into the coil, a current is started or stopped in a neighbouring circuit, a transient current is set up in the first circuit. This transient current lasts only as long as the value of the current in the neighbouring circuit is altering and ceases whenever the latter attains a steady value. Such currents are called induced currents and their laws were determined experimentally by Faraday.

In both the above cases the magnitude of the induced current depends on the resistance of the circuit, *i.e.* on the material of which the wire is composed. The induced electromotive force, that is the resistance of the circuit multiplied by the current, is the same no matter what the material of the circuit is. We shall now proceed

to give the mathematical expression for the induced electromotive force. This was first obtained by F. E. Neumann in 1845, but from a different standpoint.

At a point distant r from a magnetic pole of strength m, the medium being air, the magnetic intensity H is given by m/r^2. Just as in the analogous case of the electric charge, the medium in the field of the magnetic pole is supposed to be strained. This strain is specified at any point by B, the magnetic induction at that point. B, like electric displacement, has the same value no matter what the nature of the medium in the field is. For an isotropic medium B and H have the same direction, and $\mathsf{B}=\mu\mathsf{H}$, where μ is a quantity called the magnetic permeability, μ being different for different media. If α, β, γ denote the components of H, then $\mu\alpha$, $\mu\beta$, $\mu\gamma$ denote the components of B. When the medium is air B and H have the same value. We shall confine our attention wholly to isotropic media.

Suppose now that we have an electric circuit in a magnetic field. At every point in the field B has a definite direction and magnitude. Draw any surface with the circuit as edge. At every point on this surface the direction of B is inclined to the normal to the surface. Divide the surface into elements, and multiply each element by the normal component of B at that point. Then the sum of these products taken over the whole surface may be written

$$\iint \mu(l\alpha + m\beta + n\gamma)\,d\mathsf{S},$$

where l, m, n give the direction cosines of the normal to $d\mathsf{S}$. The integral is consequently the surface integral of normal magnetic induction through the circuit.

If a magnet is moved near the circuit or if a current is started in a second circuit in its neighbourhood, the value of the integral undergoes a change and a current is induced in the first circuit. The induced electromotive force is proportional to the rate of change of the integral. If E denotes the electric intensity at any point in the first circuit, then the electromotive force acting round that circuit is given by $\int \mathsf{E}\,ds$, where the integration is taken round the circuit. The law of the induction of currents can then be stated mathematically as follows:

$$c\int \mathsf{E}\,ds = -\frac{\partial}{\partial t}\iint \mu(l\alpha + m\beta + n\gamma)\,d\mathsf{S}.$$

The minus sign means that if the surface integral of normal induction is increasing, its direction is connected with the line integral of electric intensity in the manner typified by a left-handed screw. In the usual statement of Stokes' theorem, the curl and line integral of the vector are connected in the manner typified by a right-handed screw (cf. § 53).

In the above equation E is measured in electrostatic units and c is the constant of proportionality, which has the same value as in the

mathematical expression of Ampère's theorem. On the electromagnetic system of units the unit of electromotive force is chosen so as to make c unity. The equation thus gives us a means of defining the electromagnetic unit of electromotive force. The equation also gives us a means of defining μ. For suppose we have a primary circuit wound uniformly round the surface of a cylinder and a secondary circuit wound round the outside of the primary near the middle of the cylinder. If the primary current is broken, a current is induced in the secondary, and the total quantity of electricity passing round the latter is proportional to the induction through the cylinder, *i.e.* to the magnetic permeability of the medium filling the cylinder.

The equation expressing the law of current induction is sometimes called the second circuital equation.

§ 120. **Currents induced in a mass of metal.**

Let a lump of soft iron be placed in a changing magnetic field. Then the magnetic induction at every point in the iron is changing. If we imagine a closed curve drawn wholly in the iron, the surface integral of normal magnetic induction taken over any surface bounded by the curve is also changing. Consequently there is an induced electromotive force round the curve. But the curve may be drawn in an infinite number of positions in the mass of metal. We are thus led to the conclusion that there is an electric intensity with a definite magnitude and direction at every point in the metal. It is not necessary that the medium in the field should be iron; we can imagine the closed curve drawn in air quite as well. We thus come to the general conclusion, that whenever there is a changing field of magnetic induction, at every point in that field there is an induced electric intensity of definite magnitude and direction at every time during the change. This electric intensity vanishes whenever the value of the magnetic induction becomes constant.

The rate of change of magnetic induction can be determined when the electric intensity is fully known. For, by Stokes' theorem,

$$c\int \mathsf{E}\, ds = c\iint\left\{l\left(\frac{\partial \mathsf{Z}}{\partial y}-\frac{\partial \mathsf{Y}}{\partial z}\right)+m\left(\frac{\partial \mathsf{X}}{\partial z}-\frac{\partial \mathsf{Z}}{\partial x}\right)+n\left(\frac{\partial \mathsf{Y}}{\partial x}-\frac{\partial \mathsf{X}}{\partial y}\right)\right\} d\mathsf{S}.$$

The expression on the left hand is c times the line integral of the electric intensity taken round any closed curve; the right hand is c times the surface integral of the normal component of the curl of the electric intensity taken over any surface bounded by the closed curve. By combining the above equation with the second circuital equation, we obtain

$$\frac{\partial}{\partial t}\iint \mu(l\alpha+m\beta+n\gamma)\, d\mathsf{S}$$
$$= -c\iint\left\{l\left(\frac{\partial \mathsf{Z}}{\partial y}-\frac{\partial \mathsf{Y}}{\partial z}\right)+m\left(\frac{\partial \mathsf{X}}{\partial z}-\frac{\partial \mathsf{Z}}{\partial x}\right)+n\left(\frac{\partial \mathsf{Y}}{\partial x}-\frac{\partial \mathsf{X}}{\partial y}\right)\right\} d\mathsf{S}.$$

And, in the same way as in deriving the first three equations for the electromagnetic field, this equation decomposes into the following three equations:

$$\frac{\mu}{c}\frac{\partial\alpha}{\partial t} = -\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right), \quad \frac{\mu}{c}\frac{\partial\beta}{\partial t} = -\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right), \quad \frac{\mu}{c}\frac{\partial\gamma}{\partial t} = -\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right).$$

These are Maxwell's second three equations for the electromagnetic field. They may be combined in the equation

$$\frac{\mu}{c}\frac{\partial \mathsf{H}}{\partial t} = -\operatorname{curl} \mathsf{E}.$$

§ 121. It has been seen that the constant of proportionality in Ampère's theorem and in the law of current induction is the same in both cases. The one law enables us to define the unit of current on the electromagnetic system and the other enables us to define the unit of electromotive force on the same system. And yet the unit of current multiplied by the unit of electromotive force should give one erg per second. The explanation is, that the law of current induction is not independent of Ampère's theorem. It was discovered experimentally but could have been foretold theoretically from the latter.

For suppose we have a linear closed circuit in which there is a battery of constant electromotive force e and that in this circuit a current i electrostatic units is flowing. Let there be a pole of strength m at a point at which the circuit subtends a solid angle Ω. Then by Ampère's theorem the potential energy of the pole is $mi\Omega/c$. The current will act on the pole and move it into a position where the potential is less. Let $mi\,d\Omega/c$ be the change in potential energy of the pole in time dt. Then $-mi\,d\Omega/c$ is work done by the battery. The rate of working of the battery is ei, but owing to the work done on the pole, the rate at which heat is produced in the circuit is only

$$ei + \frac{mi}{c}\frac{\partial\Omega}{\partial t}$$

$\left(\frac{\partial\Omega}{\partial t}\text{ is negative}\right)$. There is thus an induced electromotive force, or back E.M.F. equal to $-\frac{m}{c}\frac{\partial\Omega}{\partial t}$. The total number of lines of induction issuing from the pole is $4\pi m$ and the surface integral of normal induction through the circuit is $m\Omega$. Its rate of increase is thus $m\frac{\partial\Omega}{\partial t}$. That is, the electromotive force induced round the circuit is $-\frac{1}{c}$ times the rate of increase of the surface integral of normal induction. The second circuital theorem can thus be derived from the first circuital theorem by means of the principle of energy.

The constant c can best be determined by measuring the capacity of a condenser both in electrostatic and electromagnetic units. The dimensions of capacity are the dimensions of charge divided by potential

and the dimensions of potential are the dimensions of work divided by charge. The dimensions of capacity are therefore the dimensions of charge to the second power divided by the dimensions of work. The unit of work is of course the same on both systems and the units of quantity are to one another as the units of current. Hence the numerical value of the capacity on the electrostatic system is c^2 times its numerical value on the electromagnetic system. The numerical value on the electrostatic system can be found from the dimensions of the condenser. In the case of a sphere it is equal to the radius. The numerical value of the capacity on the electromagnetic system can be found by experiment. Hence c.

§ 122. **Electromagnetic waves.**

Let κ, the conductivity of the medium, be zero. Then the equations of the electromagnetic field become

$$\frac{k}{c}\frac{\partial X}{\partial t}=\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\quad \frac{k}{c}\frac{\partial Y}{\partial t}=\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\quad \frac{k}{c}\frac{\partial Z}{\partial t}=\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y},$$

$$-\frac{\mu}{c}\frac{\partial\alpha}{\partial t}=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},\quad -\frac{\mu}{c}\frac{\partial\beta}{\partial t}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\quad\text{and}\quad -\frac{\mu}{c}\frac{\partial\gamma}{\partial t}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}.$$

Differentiating the first with regard to t and substituting from the last two, we obtain

$$\frac{k}{c}\frac{\partial^2 X}{\partial t^2}=\frac{\partial^2\gamma}{\partial y\,\partial t}-\frac{\partial^2\beta}{\partial z\,\partial t}=-\frac{c}{\mu}\left(\frac{\partial^2 Y}{\partial x\,\partial y}-\frac{\partial^2 X}{\partial y^2}-\frac{\partial^2 X}{\partial z^2}+\frac{\partial^2 Z}{\partial x\,\partial z}\right),$$

i.e.
$$\frac{\mu k}{c^2}\frac{\partial^2 X}{\partial t^2}=\frac{\partial^2 X}{\partial x^2}+\frac{\partial^2 X}{\partial y^2}+\frac{\partial^2 X}{\partial z^2}-\frac{\partial}{\partial x}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right).$$

But $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0$, since we suppose that no charges exist in the field. Hence

$$\frac{\mu k}{c^2}\frac{\partial^2 X}{\partial t^2}=\frac{\partial^2 X}{\partial x^2}+\frac{\partial^2 X}{\partial y^2}+\frac{\partial^2 X}{\partial z^2}.$$

This equation states that X is propagated by wave motion, the velocity of the waves being $c/\sqrt{\mu k}$. We can prove the same for Y, Z, α, β, γ by proceeding in exactly the same way. Hence the electric and magnetic intensities are propagated in a dielectric with velocity $c/\sqrt{\mu k}$. In the case of air $\mu=k=1$. The numerical value of c as found in the laboratory by the method described in § 121 is $3\ 10^{10}$ cms./sec., and this coincides with the velocity of light in air. We thus draw the conclusion, that light is an electromagnetic wave.

This striking result was first published by Maxwell in 1865. His theory of the electromagnetic field was not generally accepted until Hertz performed his experiments in 1887-88, because, previous to that time, there were no experiments that could be explained only by it.

§ 123. **Hertz's experiments.**

If the inside and outside of a Leyden jar are connected by a coil of wire possessing self-induction and having a spark gap at one point in its length, and if the jar is charged by means of an influence machine, the difference of potential at the two sides of the gap eventually reaches a value when the dielectric resistance of the air breaks down and the jar discharges across the gap producing a bright spark. If the resistance of the wire is not too great and this spark is examined with a rotating mirror, it is seen to consist of three or four sparks alternately in different directions. Oscillations are set up; the current does not merely pass from the positive to the negative side, ceasing when the original difference of potential is annulled, but it continues until a difference of potential is produced in the reverse direction, the side which originally possessed the negative charge now having the positive charge. Then the direction of the current reverses, and we have the electricity oscillating from the one side to the other until finally the heat of the current is no longer sufficient to maintain the air of the gap in a conducting state and the circuit is broken. The original electrostatic energy is dissipated in heat in the wire and in the spark gap.

Here then we have changing conduction currents, and from what has been said above we would expect displacement currents to be produced. In the theory of the discharge of a condenser which was given by Lord Kelvin in 1853, only the conduction current was considered, and Lord Kelvin's theory has stood the test of experiment well. The reason why displacement currents are not in evidence during the ordinary oscillatory discharge of a Leyden jar is, that the oscillations are not fast enough and that the circuit is too "closed." The displacement is all in the glass of the condenser and does not radiate out into the field.

According to the theory the period of discharge of a condenser is given by $2\pi\sqrt{LC}$, where L is the self-induction of the circuit and C is the capacity of the condenser. In Feddersen's experiments proving the theory, the average period was 10^{-5} secs. By reducing the capacity and induction so as to bring the period down to about 10^{-8} secs. and by altering the form of the circuit, Hertz was able to show the existence of displacement currents and to prove conclusively that the dielectric played a part in the discharge.

Hertz employed a vibrator consisting of two spheres connected by a straight rod with a spark gap in its middle, the two sides of the spark gap being connected with the secondary of an induction coil. When the primary of the induction coil was broken, vibrations of a large period were set up in the secondary, and the spheres became charged with electricity of different sign, until the resistance of the spark gap broke down. Then they discharged across the gap and the electricity surged backwards and forwards between the two spheres, until the gap ceased to conduct. The number of complete vibrations was not large,

about three or four, and they were not isochronous, because the resistance of the gap was large and was changing all the time.

The vibrator may be considered apart from the induction coil; it cannot discharge back through the secondary of the coil owing to the great inductance of the latter. Waves are sent out from the vibrator, and Hertz demonstrated their presence in the room by means of a receiver consisting of a circle of wire with a micrometer gap in it. When the circle was placed so that the magnetic intensity normal to its plane was altering, sparks passed across the gap, and the sparking distance gave a means of estimating roughly the intensity of the field.

§ 124. **Hertz's theory of the electric doublet.**

An electric doublet is a system of two equal and opposite electrostatic charges a constant small distance apart. The product of either charge and the distance between them gives the moment of the doublet.

The following four equations hold in the field of a Hertzian vibrator:

$$\frac{k}{c}\frac{\partial \mathsf{E}}{\partial t} = \text{curl}\,\mathsf{H}, \quad \text{......(1)}$$

$$\frac{\mu}{c}\frac{\partial \mathsf{H}}{\partial t} = -\text{curl}\,\mathsf{E}, \quad \text{......(2)}$$

$$\text{div}\,\mathsf{E} = 0, \quad \text{......(3)}$$

$$\text{div}\,\mathsf{H} = 0. \quad \text{......(4)}$$

The last two equations merely express the condition that there are no electrostatic charges or magnets in the field. In order to find the state of affairs in the field we should have to solve the above four equations together with the appropriate boundary conditions on the surface of the vibrator and at infinity. It is, however, impossible to express the conditions for the surface of the vibrator mathematically, and a rigorous solution is beyond us. Hertz succeeded in obtaining an approximate solution, which agrees with the result of experiment, and which is of very great interest from its analogy with the mechanism of light production.

In order to obtain Hertz's solution, we suppose that the vibrator is replaced by an electric doublet, the moment of which varies harmonically with the time. This is an approximation to the state of affairs on the vibrator. Take the centre of this doublet as origin and its axis as Oz. Call all the planes through the axis meridian planes. Then by symmetry the lines of magnetic intensity are everywhere circles round the axis, and the electric intensity at every point lies in a meridian plane.

Starting from equation (4), we have, since $\gamma = 0$,

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} = 0.$$

This is the condition that $\alpha\,dy - \beta\,dx$ is a complete differential of some function of x, y. Write $\frac{\partial\Pi}{\partial t}$ for this function. Then

$$\alpha = \frac{\partial^2\Pi}{\partial t\,\partial y}, \quad \beta = -\frac{\partial^2\Pi}{\partial t\,\partial x}.$$

Equation (1) may be written as

$$\frac{k}{c}\frac{\partial \mathsf{X}}{\partial t} = -\frac{\partial\beta}{\partial z} = \frac{\partial^3\Pi}{\partial t\,\partial x\,\partial z}, \qquad \frac{k}{c}\frac{\partial \mathsf{Y}}{\partial t} = \frac{\partial\alpha}{\partial z} = \frac{\partial^3\Pi}{\partial t\,\partial y\,\partial z},$$

$$\frac{k}{c}\frac{\partial \mathsf{Z}}{\partial t} = \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y} = -\frac{\partial^3\Pi}{\partial t\,\partial x^2} - \frac{\partial^3\Pi}{\partial t\,\partial y^2}.$$

From this we obtain

$$\frac{\partial}{\partial t}\left(\frac{k\mathsf{X}}{c} - \frac{\partial^2\Pi}{\partial x\,\partial z}\right) = 0, \quad \frac{\partial}{\partial t}\left(\frac{k\mathsf{Y}}{c} - \frac{\partial^2\Pi}{\partial y\,\partial z}\right) = 0, \quad \frac{\partial}{\partial t}\left(\frac{k\mathsf{Z}}{c} + \frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right) = 0.$$

From the conditions of the problem there can be no part of X, Y or Z independent of t; hence, on integrating the above three equations, the constants of integration are each zero. We have therefore

$$k\mathsf{X} = c\frac{\partial^2\Pi}{\partial x\,\partial z}, \qquad k\mathsf{Y} = c\frac{\partial^2\Pi}{\partial y\,\partial z}, \qquad k\mathsf{Z} = -c\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right).$$

The first two components of equation (2) are

$$-\frac{\mu}{c}\frac{\partial\alpha}{\partial t} = \frac{\partial \mathsf{Z}}{\partial y} - \frac{\partial \mathsf{Y}}{\partial z}, \qquad -\frac{\mu}{c}\frac{\partial\beta}{\partial t} = \frac{\partial \mathsf{X}}{\partial z} - \frac{\partial \mathsf{Z}}{\partial x}.$$

Substituting the values for α, β, X, Y, Z, which we have already found, we obtain

$$-\frac{\mu}{c}\frac{\partial^3\Pi}{\partial t^2\partial y} = -\frac{c}{k}\frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right) - \frac{c}{k}\frac{\partial^3\Pi}{\partial y\,\partial z^2}$$

$$+\frac{\mu}{c}\frac{\partial^3\Pi}{\partial t^2\partial x} = \frac{c}{k}\frac{\partial^3\Pi}{\partial x\,\partial z^2} + \frac{c}{k}\frac{\partial}{\partial x}\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right).$$

These may be written

$$\frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial t^2} - \frac{c^2}{k\mu}\nabla^2\Pi\right) = 0, \quad \frac{\partial}{\partial x}\left(\frac{\partial^2\Pi}{\partial t^2} - \frac{c^2}{k\mu}\nabla^2\Pi\right) = 0,$$

which give on integration

$$\frac{\partial^2\Pi}{\partial t^2} = \frac{c^2}{k\mu}\nabla^2\Pi + f(z, t).$$

We can put $f(z, t) = 0$ without loss of generality because the effect of $f(z, t)$ is merely to add to the expression for Π a term independent of x and y.* The five quantities we are concerned with, α, β, X, Y, Z,

* This may most easily be seen by adding $\psi(z, t)$ to Π and substituting in the equation. The substitution of the new term gives a function of z and t, $\chi(z, t)$, which can be arranged to remove the original $f(z, t)$.

are all obtained by processes involving differentiation of Π with respect to x or y, and hence are independent of this additional term.

Since the disturbance is radiated out from the origin, the general solution must be of the form

$$\Pi = \frac{1}{r}\{f_1(r - vt) + f_2(r + vt)\},$$

where $v = c/\sqrt{k\mu}$. A solution adapted to the doublet is

$$\Pi = \frac{\phi}{r}\sin(mr - nt),$$

where $n/m = v$.

For consider points close enough to the origin for mr to be small in comparison with 2π. This means only that r must be small in comparison with λ. Then

$$\Pi = -\frac{\phi}{r}\sin nt.$$

But $k\mathsf{X} = c\dfrac{\partial^2\Pi}{\partial x\,\partial z}$, $k\mathsf{Y} = c\dfrac{\partial^2\Pi}{\partial y\,\partial z}$, and in this case $k\mathsf{Z} = c\dfrac{\partial^2\Pi}{\partial z^2}$ since $\dfrac{1}{r}$ satisfies the equation $\dfrac{\partial^2\Pi}{\partial x^2} + \dfrac{\partial^2\Pi}{\partial y^2} + \dfrac{\partial^2\Pi}{\partial z^2} = 0$. Now

$$\frac{\partial\Pi}{\partial z} = -\phi\sin nt\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = +\frac{\phi\sin nt}{r^2}\frac{z}{r} \quad \text{or} \quad \frac{\phi\sin nt}{r^2}\cos\theta,$$

if θ is the angle r makes with the z-axis. At points considerably less than a wave-length distant from the origin, then, the components of electric intensity X, Y, Z are obtained by differentiating $\dfrac{c\phi\sin nt\cos\theta}{kr^2}$ with respect to x, y and z. But, by analogy with the small magnet, $\dfrac{c\phi\sin nt\cos\theta}{kr^2}$ is the expression for the potential due to an electric doublet of moment $c\phi\sin nt$ in a medium of specific inductive capacity k. At distances great in comparison with the length of the doublet but small in comparison with the wave-length, Hertz's solution thus gives the values of the electric intensity which we would expect to get.

Similarly with the magnetic intensity. The magnetic intensity in the case of a current γ flowing in a closed conductor can be calculated correctly on the assumption that each length ds contributes a part $\dfrac{\gamma\sin\theta\,ds}{cr^2}$ at a point distant r from it, γ being expressed in electrostatic units, r making an angle θ with ds and the direction of this part being perpendicular to the plane containing r and ds. Apply this to the doublet; $\gamma\,ds$ has the value $cn\phi\cos nt$, since this expression is obtained by differentiating the moment with respect to the time. The expression for H is therefore

$$\frac{n\phi\cos nt\sin\theta}{r^2}.$$

From the value of Π, we have

$$\alpha = \frac{\partial^2 \Pi}{\partial t\, \partial y} = \frac{\partial^2}{\partial t\, \partial y}\left\{-\frac{\phi}{r}\sin nt\right\} = \frac{n\phi \cos nt}{r^2}\frac{\partial r}{\partial y}.$$

We know that the lines of magnetic intensity are circles. Suppose that the y-axis is in the meridian at the point considered. Then $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$, where θ is defined as before, and β becomes zero. The two values for the magnetic intensity therefore agree.

Also it is obvious that α, β, X, Y, Z are zero at infinity. Hence the solution satisfies the boundary conditions.

In differentiating $\frac{\phi}{r}\sin(mr - nt)$ with respect to x, y and z, we obtain two types of terms, those with $1/r^2$ as factor and those with $1/r$ as factor. At large distances from the origin the terms of the first type may be neglected. Consider a wave at a large distance from the origin and choose the yz-plane so as to contain the direction of propagation of the wave. Then $r = \sqrt{y^2 + z^2}$,

$$k\mathsf{Y} = -\frac{c\phi m^2}{r}\sin(mr - nt)\sin\theta\cos\theta, \quad k\mathsf{Z} = \frac{c\phi m^2}{r}\sin(mr - nt)\sin^2\theta$$

and

$$\alpha = \frac{\phi mn}{r}\sin(mr - nt)\sin\theta.$$

Hence $\mathsf{Z}\cos\theta + \mathsf{Y}\sin\theta$ is equal to zero and the electric intensity is perpendicular to r. At large distances from the vibrator, therefore, we have the electric and magnetic intensities both perpendicular to the direction of propagation and to one another.

§ 125. Poynting's theorem.

If a conductor receives an electrostatic charge, the energy of the charge is stored up in the field. This can be shown very well with a Leyden jar, the inner and outer coatings of which can be detached from the glass. If the condenser is insulated and charged, and if it is taken apart with insulating tongs and the two coatings put into contact with one another, no spark passes between them. But if it is put together again and then discharged, the spark is as great as it would have been had the condenser never been taken apart. The energy of the charge has apparently been stored up in the glass.

The energy of a system of charged conductors can, of course, be calculated from the work done in bringing each elementary charge from infinity, in analogy with the method of calculating the potential energy of a system of gravitating masses. It is found that the same numerical value can always be obtained by assuming that there is an amount of electrostatic energy stored at every point of the field equal to

$$\frac{k\mathsf{E}^2}{8\pi} \quad \text{or} \quad \frac{\mathsf{ED}}{2}$$

per unit volume, E and D being the electric intensity and displacement at the point. The assumption is therefore taken to be correct. The expression $\mathsf{ED}/2$ brings out the meaning of displacement very clearly owing to its analogy with the formula "half tension by extension" for the work done in stretching a spiral spring.

Similarly, at a point in a magnetic field, where H and B are respectively the magnetic intensity and induction, we assume that there is a quantity of energy $\frac{\mu \mathsf{H}^2}{8\pi}$ per unit volume. This assumption gives the same value for the energy of a system of electric circuits as is obtained by using the equivalence of each circuit to a magnetic shell.

We assume, therefore, that the density of the total energy in the field is given by

$$\frac{1}{8\pi}(k\mathsf{E}^2 + \mu \mathsf{H}^2).$$

Suppose now that we have a certain region of space bounded by a closed surface. The energy in this region is given by

$$\iiint \frac{1}{8\pi}(k\mathsf{E}^2 + \mu\mathsf{H}^2)\,dx\,dy\,dz$$
$$= \iiint \frac{1}{8\pi}\{k(\mathsf{X}^2 + \mathsf{Y}^2 + \mathsf{Z}^2) + \mu(\alpha^2 + \beta^2 + \gamma^2)\}\,dx\,dy\,dz,$$

the integration being taken throughout the whole region. The rate of increase of the energy in the region is obtained by differentiating the integral with respect to t, and is equal to

$$\iiint \frac{1}{4\pi}\left\{k\left(\mathsf{X}\frac{\partial \mathsf{X}}{\partial t} + \mathsf{Y}\frac{\partial \mathsf{Y}}{\partial t} + \mathsf{Z}\frac{\partial \mathsf{Z}}{\partial t}\right) + \mu\left(\alpha\frac{\partial \alpha}{\partial t} + \beta\frac{\partial \beta}{\partial t} + \gamma\frac{\partial \gamma}{\partial t}\right)\right\}dx\,dy\,dz.$$

Substituting for $k\frac{\partial \mathsf{X}}{\partial t}$, ..., ..., $\mu\frac{\partial \alpha}{\partial t}$, ..., ..., from the equations of the electromagnetic field, this becomes

$$\frac{c}{4\pi}\iiint\left\{\Sigma\mathsf{X}\left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}\right) - \Sigma\alpha\left(\frac{\partial \mathsf{Z}}{\partial y} - \frac{\partial \mathsf{Y}}{\partial z}\right)\right\}dx\,dy\,dz$$
$$= \frac{c}{4\pi}\iiint\left\{\frac{\partial}{\partial x}(\beta\mathsf{Z} - \gamma\mathsf{Y}) + \frac{\partial}{\partial y}(\gamma\mathsf{X} - \alpha\mathsf{Z}) + \frac{\partial}{\partial z}(\alpha\mathsf{Y} - \beta\mathsf{X})\right\}dx\,dy\,dz$$
$$= \frac{c}{4\pi}\iint\{l(\beta\mathsf{Z} - \gamma\mathsf{Y}) + m(\gamma\mathsf{X} - \alpha\mathsf{Z}) + n(\alpha\mathsf{Y} - \beta\mathsf{X})\}\,d\mathsf{S},$$

by Gauss's theorem. The vector, the components of which are $\beta\mathsf{Z} - \gamma\mathsf{Y}$, $\gamma\mathsf{X} - \alpha\mathsf{Z}$ and $\alpha\mathsf{Y} - \beta\mathsf{X}$, is evidently at right angles to both H and E, and its numerical value is equal to

$$\{(\beta\mathsf{Z} - \gamma\mathsf{Y})^2 + (\gamma\mathsf{X} - \alpha\mathsf{Z})^2 + (\alpha\mathsf{Y} - \beta\mathsf{X})^2\}^{\frac{1}{2}} = \mathsf{EH}\sin\theta,$$

where θ is the angle between the directions of H and E.

The surface integral is the surface integral of the normal component of $\frac{c\mathsf{EH}\sin\theta}{4\pi}$ taken over the surface bounding the region. It is natural then to interpret $\frac{c\mathsf{EH}\sin\theta}{4\pi}$ at a point in space as the rate of flow of energy per unit area at that point. This result is due to Professor Poynting.

§ 126. Application of Poynting's theorem.

Let us apply the theorem to the case of a long straight homogeneous cylindrical wire of circular section carrying a steady current γ. Let r be the radius of the wire.

Consider the portion of the wire intercepted between two planes perpendicular to its axis and distant d apart. Let R be the total resistance of this portion. Let us consider the rate at which energy is flowing into this portion. We have to form the expression $\frac{c\mathsf{EH}\sin\theta}{4\pi}$ over the side and ends of a cylinder of length d and radius r.

The lines of magnetic intensity are circles and on the surface of the cylinder H has the value $2\gamma/(cr)$. The lines of electric intensity inside the cylinder are straight lines parallel to its axis. The total difference of potential between the two ends of the cylinder is $\gamma\mathsf{R}$; hence the electric intensity is $\gamma\mathsf{R}/d$. It is everywhere at right angles to H. The direction of $\frac{c\mathsf{EH}\sin\theta}{4\pi}$ at the ends of the cylinder is parallel to these ends; hence no energy enters into the cylinder through the ends. On the surface it is perpendicular to the surface and has the value $\frac{\gamma^2\mathsf{R}}{2\pi rd}$. If we multiply this by the total area of the curved surface, we find for the rate at which energy is flowing into the cylinder

$$2\pi rd \times \frac{\gamma^2\mathsf{R}}{2\pi rd} = \gamma^2\mathsf{R},$$

and that of course is the rate at which heat is being produced in the cylinder.

If the current is produced by an electric battery, chemical energy is converted into electromagnetic energy in the battery and flows through the dielectric to the wire where it is converted into heat. It does not flow along the wire although the latter guides its flow through the dielectric.

If an alternating current flows along the wire, periodic waves are therefore propagated in from the surface, the amplitude decreasing with the distance from the latter. We would thus expect the current to be denser near the surface of the wire. This result is borne out by experiment. If κ, the conductivity, is very great in comparison with k, we find that the equations for the propagation of H and E are of the same form as the equation for the conduction of heat.

§ 127. **Propagation of a plane wave.**

Consider the expression $\mathsf{Y} = \mathsf{B} \cos \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right)$.

It represents a plane wave of electric intensity propagated in the direction of the positive x-axis, v being the velocity of the wave, τ its period and B the maximum value of its amplitude. In any plane parallel to yz at any time the electric intensity has everywhere the same value. If we fix our attention on a fixed plane, then, as time progresses, the electric intensity undergoes a simple harmonic variation. If we fix our attention on a definite time and move the plane instantaneously in the direction of the x-axis, then the electric intensity again undergoes a simple harmonic variation when regarded as a function of the distance. Its direction, however, always remains parallel to the y-axis.

Put $\mathsf{X} = \mathsf{Z} = 0$ and substitute for X, Y and Z in the second three equations of the electromagnetic field. Then

$$-\frac{\mu}{c}\frac{\partial \alpha}{\partial t} = 0, \quad -\frac{\mu}{c}\frac{\partial \beta}{\partial t} = 0, \quad -\frac{\mu}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial \mathsf{Y}}{\partial x} = \mathsf{B}\frac{2\pi}{\tau v} \sin \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right).$$

The constants of integration must be zero, as there are supposed to be no permanent magnets or steady currents in the field. We thus obtain

$$\alpha = \beta = 0, \quad \gamma = \mathsf{B}\frac{c}{\mu v} \cos \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right).$$

This represents a plane wave of magnetic intensity, of the same period and velocity as the former wave propagated in the same direction, the magnetic intensity in the wave being always parallel to the z-axis. According to the equations of the electromagnetic field, we cannot have the one wave without the other. Both together are said to constitute a plane electromagnetic wave plane polarised in the zx-plane. In the wave the electric and magnetic intensities are at right angles both to one another and to the direction of propagation.

Similarly, if we had started out with the wave

$$\mathsf{Z} = \mathsf{C} \cos \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right),$$

by substitution in the equations of the electromagnetic field we would have found associated with it the wave

$$\beta = -\mathsf{C}\frac{c}{\mu v} \cos \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right).$$

Together they constitute a plane electromagnetic wave plane polarised in the xy-plane.

Suppose that the plane polarised wave is not propagated in the direction of one of the coordinate axes but in any direction whatever, the

direction cosines of which are l, m, n. Then it may be represented by

$$X = A\cos\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right), \qquad Y = B\cos\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right),$$

$$Z = C\cos\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right).$$

Now
$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0.$$

Substituting in this equation and cancelling out the common factor

$$\frac{2\pi}{\tau v}\sin\left(t - \frac{lx+my+nz}{v}\right),$$

we obtain $lA + mB + nC = 0$, *i.e.* A, B and C are not independent, but the resultant electric intensity must be at right angles to the direction of propagation.

Substituting in the second three equations of the electromagnetic field, we obtain

$$-\frac{\mu}{c}\frac{\partial\alpha}{\partial t} = (mC - nB)\frac{2\pi}{\tau v}\sin\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right),$$

$$-\frac{\mu}{c}\frac{\partial\beta}{\partial t} = (nA - lC)\frac{2\pi}{\tau v}\sin\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right),$$

$$-\frac{\mu}{c}\frac{\partial\gamma}{\partial t} = (lB - mA)\frac{2\pi}{\tau v}\sin\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right),$$

whence
$$\alpha = (mC - nB)\frac{c}{\mu v}\cos\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right),$$

$$\beta = (nA - lC)\frac{c}{\mu v}\cos\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right),$$

$$\gamma = (lB - mA)\frac{c}{\mu v}\cos\frac{2\pi}{\tau}\left(t - \frac{lx+my+nz}{v}\right).$$

We see from the form of the coefficients that (α, β, γ) is at right angles to both (X, Y, Z) and (l, m, n).

The velocity of the wave is given by $c/\sqrt{k\mu}$. It is found that as far as light waves are concerned, $\mu = 1$. Also we know that the velocity of a light wave is given by c/n, where n is the index of refraction of the medium for the particular colour in question. We must therefore have $\sqrt{k} = n$ or $k = n^2$. Hence k cannot be a constant as far as light vibrations are concerned, but must depend on the frequency of the vibrations, although in electrostatics it was a constant for any one medium.

In what follows we shall suppose we are dealing with monochromatic light.

§ 128. **Energy of a plane wave.**

Suppose that the wave is propagated in the x direction and that it is polarised in the zx-plane. Then it may be represented by

$$\mathsf{Y} = \mathsf{B} \cos \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right), \qquad \gamma = \mathsf{B} \frac{c}{\mu v} \cos \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right).$$

By Poynting's theorem the energy is flowing in the direction of the x-axis, *i.e.* the direction of the flow of energy is identified with the ray, and the rate of flow at any time for any value of x is given by

$$\frac{c\mathsf{EH}\sin\theta}{4\pi} = \frac{c^2}{4\pi\mu v}\mathsf{B}^2 \cos^2 \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right) \text{ ergs/sq. cm., sec.}$$

This expression oscillates between zero and a constant positive value, but never changes sign. The energy flow is therefore always forward. The period of the oscillations is so small that they cannot be detected by the eye or any physical instrument; it is the mean value that is important. Now the mean value of $\cos^2\theta$, between $\theta = 0$ and $\theta = \pi$, is $\frac{1}{2}$. Hence the intensity of the wave is equal to

$$\frac{c^2}{8\pi\mu v}\mathsf{B}^2.$$

As all our observations on light are made in air, for all practical purposes we may put $\mu v = c$. The intensity of the wave is therefore proportional to the square of the amplitude, a result which might have been derived by analogy from hydrodynamical and other considerations.

§ 129. **Boundary conditions.**

It is now necessary to determine the conditions that must be fulfilled at the boundary of two media when an electromagnetic wave passes from the one to the other. To fix our ideas, let the xy-plane be the boundary, let the specific inductive capacity of the upper medium be k, of the lower medium k', and take the axis of z positive downwards. We shall also suppose that as we pass through the boundary, the specific inductive capacity changes discontinuously from the value k to k'.

Consider the rectangle ABCD, the side AB of which is in the one medium and the side CD in the other, both AB and CD being extremely close to Ox. Let a unit magnetic pole be carried round this rectangle. Then the work done against the field must be zero, because the area of the rectangle is so small, that the displacement current flowing through it may be neglected. The work done on the ends AD and BC may be neglected owing to their being so small. Thus the work done on AB must be equal and opposite to the work done on CD, or, in other words, the magnetic intensities along AB and DC are equal. We arrive therefore at the condition

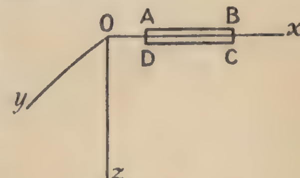

Fig. 64.

that the tangential component of magnetic intensity must have the same magnitude and direction on both sides of the boundary, that is, in this case α and β must be the same on both sides of the boundary.

Similarly, by taking unit positive electric charge round the rectangle, it may be shown that X and Y have the same value on both sides of the boundary.

Suppose now, that instead of representing a rectangular circuit, ABCD is a section of a flat right circular cylinder, the axis of which is parallel to Oz, AB and CD being sections of the ends of this cylinder. Let the area of the ends be a. Take the surface integral of the normal component of electric displacement over the surface of the cylinder. Then the part contributed by the side may be neglected owing to the area of the side being so small. The part contributed by the upper end is $-\dfrac{k\mathsf{Z}a}{4\pi}$ and the part contributed by the lower end $+\dfrac{k'\mathsf{Z}'a}{4\pi}$, where Z' is the value of Z in the second medium measured downwards. Since there is no electric charge within the cylinder, the whole integral must be zero. Hence $-\dfrac{k\mathsf{Z}a}{4\pi}+\dfrac{k'\mathsf{Z}'a}{4\pi}=0$ or $k\mathsf{Z}=k'\mathsf{Z}'$. We thus arrive at the condition that the normal component of electric displacement is the same in both media. Similarly it may be shown that the normal component of magnetic induction is the same in both media.

§ 130. **Reflection and refraction.**

Let a plane polarised plane wave of monochromatic light fall upon the plane boundary of two transparent media. Take the axis of z positive downwards and let the boundary of the two media be given by $z=0$. Let the plane of incidence be the zx-plane and let the angle of incidence be ϕ. Let the specific inductive capacity of the upper medium be k and of the lower medium k'. As is usual in problems in optics, we put the magnetic permeability of both media equal to unity.

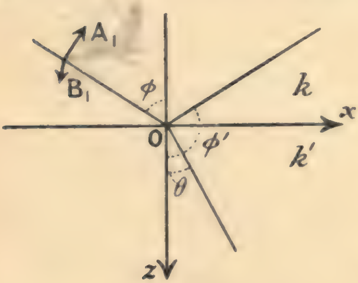

FIG. 65.

Resolve the electric intensity in the incident wave into two components, of maximum amplitude A_1 in the plane of incidence and B_1 perpendicular to the plane of incidence. Then the plane of polarisation of the incident light makes an angle $\cot^{-1}\mathsf{B}_1/\mathsf{A}_1$ with the zx-plane.

Resolve A_1 into components $\mathsf{A}_1\cos\phi$ parallel to Ox and $-\mathsf{A}_1\sin\phi$ parallel to Oz. Then the electric intensity in the incident wave may be written

$$\mathsf{X}_1=\mathsf{A}_1\cos\phi\cos\frac{2\pi}{\tau}\left(t-\frac{\sqrt{k}\{x\sin\phi+z\cos\phi\}}{c}\right),$$

$$\mathsf{Y}_1 = \mathsf{B}_1 \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x \sin \phi + z \cos \phi\}}{c}\right),$$

$$\mathsf{Z}_1 = -\mathsf{A}_1 \sin \phi \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x \sin \phi + z \cos \phi\}}{c}\right),$$

since the velocity in air is c and the direction cosines of the normal to the wave front are $\sin \phi$, 0, $\cos \phi$.

To find the magnetic intensity associated with this electric intensity, substitute for X_1, Y_1, Z_1 in the second three equations of the electromagnetic field and solve for α_1, β_1, γ_1, making the constants of integration zero. Then we obtain

$$\alpha_1 = -\mathsf{B}_1\sqrt{k} \cos \phi \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x \sin \phi + z \cos \phi\}}{c}\right),$$

$$\beta_1 = +\mathsf{A}_1\sqrt{k} \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x \sin \phi + z \cos \phi\}}{c}\right),$$

$$\gamma_1 = +\mathsf{B}_1\sqrt{k} \sin \phi \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x \sin \phi + z \cos \phi\}}{c}\right).$$

The above six equations represent the whole incident wave. When it arrives at the boundary it gives rise to a refracted and a reflected wave. We shall assume that the maximum values of the electric intensity of the refracted wave are respectively A_2 for the component in, and B_2 for the component perpendicular to the plane of incidence. We then obtain the following equations for the refracted wave simply by substituting for A_1, B_1, k and ϕ:

$$\mathsf{X}_2 = \mathsf{A}_2 \cos \theta \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k'}\{x \sin \theta + z \cos \theta\}}{c}\right),$$

$$\mathsf{Y}_2 = \mathsf{B}_2 \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k'}\{x \sin \theta + z \cos \theta\}}{c}\right),$$

$$\mathsf{Z}_2 = -\mathsf{A}_2 \sin \theta \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k'}\{x \sin \theta + z \cos \theta\}}{c}\right),$$

$$\alpha_2 = -\mathsf{B}_2\sqrt{k'} \cos \theta \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k'}\{x \sin \theta + z \cos \theta\}}{c}\right),$$

$$\beta_2 = +\mathsf{A}_2\sqrt{k'} \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k'}(x \sin \theta + z \cos \theta\}}{c}\right),$$

$$\gamma_2 = +\mathsf{B}_2\sqrt{k'} \sin \theta \cos \frac{2\pi}{\tau}\left(t - \frac{\sqrt{k'}\{x \sin \theta + z \cos \theta\}}{c}\right).$$

Similarly, for the reflected wave, we obtain

$$X_3 = A_3 \cos\phi' \cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi' + z\cos\phi'\}}{c}\right),$$

$$Y_3 = B_3 \cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi' + z\cos\phi'\}}{c}\right),$$

$$Z_3 = -A_3 \sin\phi' \cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi' + z\cos\phi'\}}{c}\right),$$

$$\alpha_3 = -B_3\sqrt{k}\cos\phi' \cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi' + z\cos\phi'\}}{c}\right),$$

$$\beta_3 = +A_3\sqrt{k}\cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi' + z\cos\phi'\}}{c}\right),$$

$$\gamma_3 = +B_3\sqrt{k}\sin\phi' \cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi' + z\cos\phi'\}}{c}\right).$$

In the above A_3 and B_3 are put respectively equal to the components of the maximum electric intensity in and perpendicular to the plane of incidence. Also, we do not assume that ϕ', the angle of reflection, is equal to ϕ, the angle of incidence. It should be noted that ϕ' is the angle that the normal to the reflected wave front makes with the *positive* direction of Oz.

We have now to apply the boundary conditions. A_1, B_1, ϕ are known, and we wish to determine A_2, B_2, θ, A_3, B_3, ϕ'. It is at once clear that for $z=0$, all the components of electric and magnetic intensity must be proportional to the same function of x and t, *i.e.* $\sqrt{k}\sin\phi = \sqrt{k'}\sin\theta = \sqrt{k}\sin\phi'$. This equation contains the laws of refraction and reflection, for it may be written

$$\frac{\sin\phi}{\sin\theta} = \sqrt{\frac{k'}{k}} = n, \quad \phi = \pi - \phi'.$$

The laws of reflection and refraction are thus derivable from the mere fact that there are boundary equations, and they do not depend on the particular form of the latter.

Since the tangential components of the electric and magnetic intensities are the same on both sides of the boundary, we have $X_1 + X_3 = X_2$ with three similar equations. These give

$$(A_1 - A_3)\cos\phi = A_2\cos\theta, \quad B_1 + B_3 = B_2,$$

$$(B_1 - B_3)\sqrt{k}\cos\phi = B_2\sqrt{k'}\cos\theta \quad \text{and} \quad (A_1 + A_3)\sqrt{k} = A_2\sqrt{k'}.$$

We have thus four equations for the four unknown quantities, and conclude that the other two boundary conditions are not independent.

This conclusion may be verified by trial in the present case. On solving the equations, we obtain

$$\mathsf{A}_3=\mathsf{A}_1\frac{\sqrt{k'}\cos\phi-\sqrt{k}\cos\theta}{\sqrt{k'}\cos\phi+\sqrt{k}\cos\theta},\quad \mathsf{B}_3=\mathsf{B}_1\frac{\sqrt{k}\cos\phi-\sqrt{k'}\cos\theta}{\sqrt{k}\cos\phi+\sqrt{k'}\cos\theta},$$

$$\mathsf{A}_2=\mathsf{A}_1\frac{2\sqrt{k}\cos\phi}{\sqrt{k}\cos\theta+\sqrt{k'}\cos\phi},\quad \mathsf{B}_2=\mathsf{B}_1\frac{2\sqrt{k}\cos\phi}{\sqrt{k'}\cos\theta+\sqrt{k}\cos\phi}.$$

On substituting $\dfrac{\sin\phi}{\sin\theta}$ for $\sqrt{\dfrac{k'}{k}}$, these results become

$$\mathsf{A}_3=\mathsf{A}_1\frac{\tan(\phi-\theta)}{\tan(\phi+\theta)},\qquad \mathsf{B}_3=-\mathsf{B}_1\frac{\sin(\phi-\theta)}{\sin(\phi+\theta)},$$

$$\mathsf{A}_2=\mathsf{A}_1\frac{2\sin\theta\cos\phi}{\sin(\phi+\theta)\cos(\phi-\theta)},\quad \mathsf{B}_2=\mathsf{B}_1\frac{2\sin\theta\cos\phi}{\sin(\phi+\theta)}.$$

The above are called Fresnel's formulae. They were first obtained by Fresnel, but not by a satisfactory method. They enable us to determine completely the reflected and refracted waves when the incident wave is known.

According to these formulae B_3 never vanishes, but A_3 becomes equal to zero when $\tan(\phi+\theta)=\infty$, *i.e.* when $\phi+\theta=\frac{\pi}{2}$. In this case $\sin\theta=\cos\phi$, and if n be put for the ratio of the refractive indices of both media, *i.e.* if $n=\sqrt{k'/k}$,

$$n=\frac{\sin\phi}{\sin\theta}=\tan\phi.$$

This value of ϕ is called the polarising angle, and this equation states Brewster's law. After reflection at this angle of incidence, natural light is plane polarised in the plane of incidence.

Fresnel's formulae can be verified very easily with a spectrometer fitted with two nicols with square ends, one attached to the collimator in front of its object glass and the other attached to the telescope in front of its object glass. These nicols can be rotated respectively about the axes of the collimator and telescope, and are provided with divided circles for reading their positions. The collimator has a circular aperture instead of a slit. From Fresnel's formulae,

$$\frac{\mathsf{B}_3}{\mathsf{A}_3}=-\frac{\mathsf{B}_1}{\mathsf{A}_1}\frac{\sin(\phi-\theta)}{\sin(\phi+\theta)}\frac{\tan(\phi+\theta)}{\tan(\phi-\theta)}=-\frac{\mathsf{B}_1}{\mathsf{A}_1}\frac{\cos(\phi-\theta)}{\cos(\phi+\theta)}.$$

$\mathsf{A}_1/\mathsf{B}_1$ is the tangent of the angle which the plane of polarisation makes with the xz plane before reflection and $\mathsf{A}_3/\mathsf{B}_3$ the tangent of the like angle after reflection. In the experiments of Jamin and Quincke, $\mathsf{A}_1/\mathsf{B}_1$ was put equal to unity, that is, the polarising nicol was set with its principal plane at 45° to the xz plane, then $\mathsf{A}_3/\mathsf{B}_3$

was determined experimentally for different values of ϕ, and the results compared with those given by the formula. The agreement was very good, only in the neighbourhood of the polarising angle was there an appreciable difference between theory and experiment. This difference has been shown to be due to the boundary conditions not being accurate. In deriving the latter, we assumed that the value of the index of refraction changed discontinuously in passing from the one medium to the other. If we assume that the change takes place gradually within a region small in comparison with the wave-length of light, we obtain more elaborate boundary conditions, and from these can derive formulae that represent the experimental results perfectly. From experiments confirming the more accurate theory, we learn that the transition layer or region in which the index of refraction changes from the one value to the other has, in the case of a polished glass surface, a thickness of about $\frac{2}{100}$ of the wave-length of sodium light.

§ 131. **Perpendicular incidence.**

In the case of perpendicular incidence ϕ and θ both become zero and Fresnel's formulae for A_3 and B_3 become indeterminate. If, however, we use the equations on page 163 immediately above Fresnel's formulae, $\cos\phi$ and $\cos\theta$ both become equal to 1, and

$$\mathsf{A}_3 = \mathsf{A}_1 \frac{\sqrt{k'} - \sqrt{k}}{\sqrt{k'} + \sqrt{k}} = \mathsf{A}_1 \frac{n-1}{n+1}, \quad \mathsf{B}_3 = \mathsf{B}_1 \frac{\sqrt{k} - \sqrt{k'}}{\sqrt{k} + \sqrt{k'}} = \mathsf{B}_1 \frac{1-n}{1+n}.$$

The fraction of the intensity reflected is therefore the same for light polarised in and perpendicular to the plane of incidence, namely $\left(\frac{n-1}{n+1}\right)^2$. In the case of reflection from glass to air, $n = 1{\cdot}5$ approximately; hence 4 % of the incident light is reflected.

§ 132. **Total reflection.**

Suppose that k' is less than k, that the wave, for example, is reflected internally at a glass-air surface. Then ϕ is the angle of incidence in the glass, θ the angle of refraction in the air and $\sin\theta = n\sin\phi$, n of course having its usual value of 1·5 or thereabouts. We have

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - n^2\sin^2\phi}.$$

Where total reflection occurs, $n^2\sin^2\phi$ is greater than 1 and $\cos\theta$ becomes imaginary. We may write it in this case,

$$\cos\theta = i\sqrt{n^2\sin^2\phi - 1}.$$

It is interesting to examine what happens to Fresnel's formulae when this imaginary value of $\cos\theta$ is substituted. Let us confine our attention to the reflected wave and examine the expression for B_3. For angles of incidence greater than the limiting angle,

$$\mathsf{B}_3 = -\mathsf{B}_1 \frac{\sin(\phi-\theta)}{\sin(\phi+\theta)} = -\mathsf{B}_1 \frac{\sin\phi\cos\theta - \cos\phi\sin\theta}{\sin\phi\cos\theta + \cos\phi\sin\theta}$$

$$= -\mathsf{B}_1 \frac{i\sin\phi\sqrt{n^2\sin^2\phi - 1} - n\sin\phi\cos\phi}{i\sin\phi\sqrt{n^2\sin^2\phi - 1} + n\sin\phi\cos\phi}$$

$$= \mathsf{B}_1 \frac{n\cos\phi - i\sqrt{n^2\sin^2\phi - 1}}{n\cos\phi + i\sqrt{n^2\sin^2\phi - 1}}.$$

On multiplying both numerator and denominator by

$$n\cos\phi - i\sqrt{n^2\sin^2\phi - 1},$$

this gives

$$\mathsf{B}_3 = \mathsf{B}_1 \frac{(n^2\cos^2\phi - n^2\sin^2\phi + 1) - 2\,in\cos\phi\sqrt{n^2\sin^2\phi - 1}}{n^2 - 1}.$$

The coefficient of B_1 is a complex quantity, the modulus of which is found by calculation to be 1 and the amplitude of which is

$$\tan^{-1}\frac{2n\cos\phi\sqrt{n^2\sin^2\phi - 1}}{n^2\cos^2\phi - n^2\sin^2\phi + 1}.$$

On writing b for the latter, the equation becomes

$$\mathsf{B}_3 = \mathsf{B}_1 e^{-ib}.$$

In order to interpret this result it is necessary to go back somewhat.

$$\mathsf{Y}_1 = \mathsf{B}_1 \cos\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi + z\cos\phi\}}{c}\right)$$

represented the electric intensity perpendicular to the plane of incidence for the incident wave. Instead of the cosine we might have written

$$\mathsf{Y}_1 = \text{real part of } \mathsf{B}_1 e^{i\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi + z\cos\phi\}}{c}\right)},$$

and we could have made similar substitutions for the other cosines. This assumption is perfectly legitimate, for the equations of the electromagnetic field and the boundary conditions are linear in X, Y, Z, α, β, γ; they are satisfied by both parts of the complex quantities taken singly, and must therefore be satisfied by their sum. Had we proceeded in this way, we should have found for Y_3 in the above case,

$$\mathsf{Y}_3 = \text{real part of } \mathsf{B}_3 e^{i\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi - z\cos\phi\}}{c}\right)}$$

$$= \text{real part of } \mathsf{B}_1 e^{i\left[\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi - z\cos\phi\}}{c}\right) - b\right]}$$

$$= \mathsf{B}_1 \cos\left[\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k}\{x\sin\phi - z\cos\phi\}}{c}\right) - b\right].$$

The amplitude of the reflected wave is therefore the same as the amplitude of the incident wave, *i.e.* no light is lost by reflection, but a phase difference is produced $= b$ and varying with the angle of incidence.

Similar results are obtained on examining the expression for A_3. Let us denote the phase difference produced in this case by a. Both components of the incident wave were originally in the same phase, but a relative phase difference has now grown up between them equal to $a - b$. The reflected wave cannot therefore be extinguished with an analysing nicol until this relative phase difference has first been removed with a compensator. The relative phase difference has been determined experimentally, and the results agree well with theory. This method of interpreting the complex amplitude is due to Fresnel.

§ 133. **Absorbing media.**

So far, in dealing with electromagnetic waves, we have confined ourselves to dielectrics. Let us now drop this restriction and assume that κ is not zero.

Then the equations of the field are

$$\frac{4\pi\kappa}{c}\mathsf{X} + \frac{k}{c}\frac{\partial \mathsf{X}}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \quad \frac{4\pi\kappa}{c}\mathsf{Y} + \frac{k}{c}\frac{\partial \mathsf{Y}}{\partial t} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \quad \frac{4\pi\kappa}{c}\mathsf{Z} + \frac{k}{c}\frac{\partial \mathsf{Z}}{\partial t} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y},$$

$$-\frac{\mu}{c}\frac{\partial \alpha}{\partial t} = \frac{\partial \mathsf{Z}}{\partial y} - \frac{\partial \mathsf{Y}}{\partial z}, \quad -\frac{\mu}{c}\frac{\partial \beta}{\partial t} = \frac{\partial \mathsf{X}}{\partial z} - \frac{\partial \mathsf{Z}}{\partial x}, \quad -\frac{\mu}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial \mathsf{Y}}{\partial x} - \frac{\partial \mathsf{X}}{\partial y}.$$

Let us assume that we are dealing with harmonic plane waves of period τ, and that exponentials are to be substituted in place of $\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \alpha, \beta, \gamma$, always on the understanding of course that the latter are the real parts of the exponentials replacing them. Then, as t occurs in every quantity in the same factor $e^{i\frac{2\pi t}{\tau}}$, dividing by $i\frac{2\pi}{\tau}$ is equivalent to integrating with respect to t, and the first of the above equations may be written

$$\frac{1}{c}(k - i2\kappa\tau)\frac{\partial \mathsf{X}}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}.$$

The second and third equations take the same form. If, as is usual in dealing with light waves, we put $\mu = 1$, the only effect of the conductivity of the medium is to replace k by the complex quantity $k - i2\kappa\tau$. The waves will therefore be represented by terms of the type,

$$\text{real part } e^{i\frac{2\pi}{\tau}\left(t - \frac{\sqrt{k - i2\kappa\tau}(lx+my+nz)}{c}\right)}.$$

On writing
$$k - i2\kappa\tau = (\mathsf{N} - i\mathsf{K})^2,$$

this becomes,
$$\text{real part } e^{i\frac{2\pi}{\tau}\left(t - \frac{(\mathsf{N} - i\mathsf{K})(lx+my+nz)}{c}\right)}$$

$$= \quad \text{,,} \quad e^{-\frac{2\pi\mathsf{K}}{\tau c}(lx+my+nz)}e^{i\frac{2\pi}{\tau}\left(t - \frac{\mathsf{N}(lx+my+nz)}{c}\right)}$$

$$= e^{-\frac{2\pi}{\tau c}(lx+my+nz)}\cos\frac{2\pi}{\tau}\left(t - \frac{\mathsf{N}(lx+my+nz)}{c}\right).$$

This represents a wave the amplitude of which diminishes as the wave advances, the energy of which is being absorbed as it progresses. The exponential factor diminishes as $lx+my+nz$ increases. In a conductor we must have therefore absorption of electromagnetic waves. The constant, which determines the absorption, has been determined for several metals for wave-lengths in the infra-red by Rubens and Hagen, and has been found to agree with the value calculated from κ. Of course the same difficulty exists in connecting up absorption of light with conductivity as in connecting up index of refraction with specific inductive capacity. The quantity of light absorbed by any substance varies with the wave-length, and the values given for the conductivity in the tables are for steady currents, *i.e.* for infinitely long waves.

EXAMPLES.

1. A copper disc is spun about an axis at right angles to its plane in a uniform magnetic field, the lines of force of which are parallel to the axis of the disc. It is touched at two points by the ends of a wire, in which is placed an electromotive force which just balances the induced electromotive force due to the rotation. Find the E.M.F. when the wires touch at any chosen points of the disc.

2. Show that in the case of the Hertzian vibrator there are longitudinal waves of electric intensity, near the origin in the direction of the axis.

3. Find an expression for the energy radiated by a Hertzian vibrator in half a period across a sphere of very large radius with its centre at the vibrator.

CHAPTER VI.

THERMODYNAMICS.

§ 134. THE science of thermodynamics is founded upon two principles. The first principle runs as follows:

When heat is transformed into work or work is transformed into heat, the quantity of heat lost or gained is proportional to the quantity of work gained or lost.

This result was founded on Joule's experiments. It is merely the principle of the conservation of energy, and is fully explained in the text-books of elementary physics.

The second principle, sometimes called the principle of entropy, is from its nature somewhat difficult to state. An account of it will be given later. Clausius has enunciated it as follows:

It is impossible for a self-acting machine, unaided by any external agency, to convey heat from one body to another at a higher temperature.

The principles of thermodynamics have been applied with success to the theory of steam engines, the radiation from an incandescent solid, the definition of temperature, the phenomena of solution, etc. Thermodynamics is not, therefore, a self-contained part of physics, but rather an aspect of the whole subject.

§ 135. Let unit mass of a gas or vapour be contained inside a cylinder of cross-sectional area A and let it be subjected to a pressure p by means of a piston. Let v be the volume of the gas.

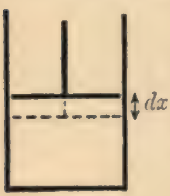

FIG. 66.

Suppose now that the piston is displaced upwards through a small distance dx. Since the displacement is small, we can assume that it does not affect the pressure appreciably. The force acting on the piston during the displacement is then $p\mathsf{A}$ and the external work done by the gas during the displacement is

$$p\mathsf{A}\,dx \quad \text{or} \quad p\,dv,$$

where dv is the increase in volume of the gas.

It is obvious that this result holds, no matter what the shape of the envelope containing the gas is. For the surface of the latter can always be divided into plane elements and, when the volume changes, each element is displaced in the direction of its normal.

We have considered p constant throughout the change of volume. If, however, the latter is large, p is a function of v. If the volume of the gas changes from v_1 to v_2, the external work done during the change is then

$$\int_{v_1}^{v_2} p\,dv.$$

§ 136. **Watt's indicator diagram.**

The state of a gas contained in a cylinder can be represented by two variables, v and p, because the temperature t is connected with p and v by means of the characteristic equation of the gas. The state of a gas can therefore be represented by a point P on a coordinate diagram, v being the abscissa and p the ordinate. Suppose that the volume and pressure of the gas change gradually, then the point will describe a curve and arrive finally at some such position as Q. The external work done by the gas during the change is

$$\int_{P}^{Q} p\,dv = \text{PQMN}.$$

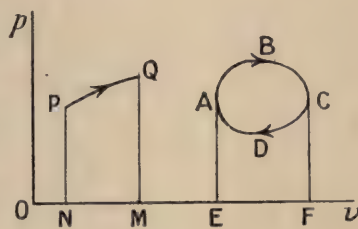

FIG. 67.

If the point representing the state of the gas is originally at A and if the volume and pressure of the gas are put through a succession of changes and finally return to their original values, the point will describe a closed curve such as ABCD, returning to the point of departure A. The gas is then said to be put through a cycle. When a system starts from a given state and returns to the same state by passing through a series of intermediate states, it is said to perform a cycle. The total external work done by the gas during the cycle in the above case is equal to the area of the closed curve ABCD, because the work done by the gas in moving from A to C is ABCFE, and the work done against the gas in moving from C to A is CDAEF.

The indicator diagram can be applied to other systems as well as to a gas contained in a cylinder, *e.g.* a wire stretched by a weight. If dx be the increment in length and F the stretching force, the work done on the wire during the change is F dx. We only require therefore to replace p and v as coordinates by means of F and x.

§ 137. The unit mass of gas in the cylinder in § 135 was supposed to be at the same temperature and density throughout. Only then can its state be truly characterised by v and p; otherwise we would require a different value of v and p for every element of its mass. Suppose now the temperature of the gas to be raised. Heat flows in from outside. While the heat is flowing in, the temperature of the gas must be unequal. Similarly, if the gas is being compressed, its pressure during compression will not be the same throughout. In

order that the working substance, the gas, may be truly homogeneous during the change, we must assume that the latter takes place infinitely slowly. We must also assume that the flow of heat from the source to the working substance does not lower the temperature of the source. Only then can the change be accurately represented on the indicator diagram.

If, during the change, the working substance receives heat from a source, the temperature of the source must be the same as that of the working substance, otherwise the change would not be infinitely slow. We can thus quite as well suppose the heat flowing in the opposite direction, from the working substance to the source. All changes can thus be effected in a reverse order. Such changes are said to be reversible, and a cycle consisting of reversible changes is a reversible cycle.

Reversible cycles and reversible changes are ideal, that is, they cannot take place in practice. We must always have a finite difference of temperature or pressure in order to produce the change. But they constitute a limiting case, which is of very great importance and at the same time very much simpler theoretically. Reversible changes in thermodynamics are somewhat analogous to dynamics with friction left out.

Unless the contrary is stated, in what follows, all changes are supposed to be made in a reversible manner.

§ 138. Consider now unit mass of a working substance and suppose a quantity of heat dq, measured in dynamical units, supplied to it. According to the first principle of thermodynamics this heat is used for two purposes:

it does external work;

it increases the intrinsic energy of the substance.

This fact may be expressed mathematically by the equation

$$dq = d\mathsf{U} + d\mathsf{W},$$

where $d\mathsf{U}$ is the increase of intrinsic energy and $d\mathsf{W}$ the external work done.

Suppose that the working substance is brought from state (1) to state (2); that the intrinsic energy changes from U_1 to U_2, that the heat supplied is q and the external work done W. Then

$$q = \mathsf{U}_2 - \mathsf{U}_1 + \mathsf{W}.$$

This change from state (1) to state (2) may be made in different ways. For each of these ways q and W may be different, but $\mathsf{U}_2 - \mathsf{U}_1$ is always the same. For, suppose that the substance is brought from state (1) to state (2) otherwise and that

$$q' = \mathsf{U}'_2 - \mathsf{U}_1 + \mathsf{W}'.$$

Then, carrying it from state (1) to state (2) by the first transformation and from state (2) to state (1) by the second transformation,

$$q - q' = U_2 - U'_2 + W - W'.$$

Since the initial state is the same as the final state, all the heat supplied must have gone into external work, *i.e.*

$$q - q' = W - W'.$$

Hence $U_2 - U'_2 = 0$, that is, the intrinsic energy depends only on the state of the substance, and is a function of the coordinates defining that state. This may be regarded as an alternative statement of the first principle of thermodynamics.

§ 139. **Carnot's cycle.**

We shall now consider a reversible cycle due to Carnot, which has played a great part in the development of thermodynamics. To make matters clearer, we shall assume in this section that the working substance is a gas, though any substance may be put through a Carnot's cycle.

The gas is contained in a cylinder D, the piston and side of which are non-conductors of heat, but the bottom of which is a perfect conductor of heat. A, B and C are three stands, A being a non-conductor of heat, but B and C having tops that conduct heat perfectly. B is kept at the constant temperature t_1 and C is kept at the constant temperature t_2, t_1 being higher than t_2.

Let the temperature of the gas in D be originally t_2. Place D on A and compress the gas by pushing the piston down. During the compression no heat is lost or gained by the gas, and the change is said to be an adiabatic or isentropic one. All the work done on the

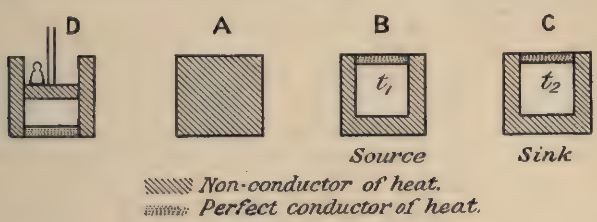

FIG. 68.

gas thus goes to raising its temperature, and we shall suppose that the temperature is increased from t_2 to t_1. If E is the point on the indicator diagram representing the initial state of the gas, and if, as the compression proceeds, the point moves along EF, EF is said to be an adiabatic curve.

When the temperature has become t_1, remove D from A, put it on B and allow the gas to expand by removing weights from the piston. During this expansion the temperature of the gas remains constant,

the expansion is said to be an isothermal one and the point in the indicator diagram moves along the isothermal curve FG. As the gas does external work during this expansion, it must receive heat from B, or the source as it is called. Let the quantity of heat received, measured in dynamical units, be q_1.

Now place D again on A and allow the gas to expand further. The expansion is an adiabatic one, and the temperature of the gas will consequently fall. Let the expansion proceed until it falls to t_2. During this expansion the point in the indicator diagram moves along the adiabatic GH.

The temperature is now the same as the initial one. Remove D from A, place it on C and push down the piston until the volume of the gas returns to its original value. This change is an isothermal one, and during it work is done on the gas. Consequently the gas must lose heat to C, and in virtue of this, C is termed the sink or condenser. Let the quantity of heat so lost be q_2, measured in dynamical units. This final change is represented on the diagram by the isothermal HE.

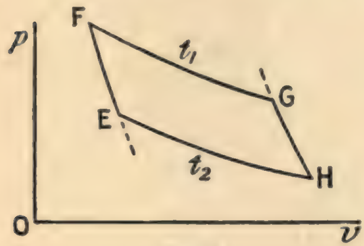

FIG. 69.

The gas has now returned to its original state, and the point representing its condition has travelled through the closed curve EFGHE. Hence the gas has done an amount of external work W equal to the area of this curve. By the first principle of thermodynamics,

$$W = q_1 - q_2.$$

The efficiency of a substance working in such a cycle, that is, the ratio of the external work done to the heat supplied from the source, is

$$\frac{W}{q_1} = \frac{q_1 - q_2}{q_1}.$$

As Carnot's cycle is a reversible one, the substance may be put through it in the reverse order in the direction EHGFE. In this case work W is done on the gas, heat q_2 is received from the sink and q_1 is given to the source. Instead of heat being converted into work, work is converted into heat.

§ 140. **Application of the second principle of thermodynamics.**

If we have two engines working between the same two temperatures t_1 and t_2 converting heat into work, one, which we shall call A, working in a reversible cycle, and the other, which we shall call B, working in an irreversible cycle, then the efficiency of B cannot be greater than that of A.

For suppose that B is more efficient than A. Let t_1 be the temperature of the source and t_2 the temperature of the sink. Let A take

a quantity of heat q_1 from the source and give up a quantity q_2 to the sink, and let B take a quantity q'_1 from the source and give up a quantity q'_2 to the sink. We can assume without loss of generality that both engines do the same amount of work in a cycle, *i.e.*

$$q_1 - q_2 = q'_1 - q'_2.$$

Also, by supposition, $$\frac{q_1 - q_2}{q_1} < \frac{q'_1 - q'_2}{q'_1}.$$

Now let engine A work backwards converting work into heat, and suppose that the work done by B is used in working A. Then to every cycle of B there corresponds a cycle of A; the resultant work done is zero, but in each cycle a quantity of heat $q_2 - q'_2$ is taken from the sink and a quantity $q_1 - q'_1$ is given to the source. Now it follows from the above equation and inequality that $q_1 > q'_1$ and $q_2 > q'_2$. Hence $q_1 - q'_1$ and $q_2 - q'_2$ are both positive, heat is being conveyed from the colder to the hotter body of the system, and no external agency is doing any work on the system. This is contrary to the second principle of thermodynamics. Consequently B cannot be more efficient than A.

The above theorem is called Carnot's principle.

It can be shown by similar reasoning that all engines working in reversible cycles between the same two temperatures have the same efficiency, for, if the less efficient engine be reversed so as to convert work into heat, and if the work done by the more efficient be employed in working it, the second principle of thermodynamics will be again infringed.

Since all engines working in reversible cycles between the same temperatures have the same efficiency, it follows that the efficiency of the Carnot cycle is independent of the working substance used. It is independent of q_2, for if q_1 be increased n times, since

$$\frac{q_1 - q_2}{q_1}$$

must remain the same, q_2 must also increase n times. It is independent of q_1, for if we have two engines working in Carnot cycles between t_1 and t_2 and the first takes n times as much heat from the source as the second, the second performs in n cycles exactly the same quantity of external work as the first and gives up exactly the same quantity of heat to the sink. The efficiency of any substance working in a Carnot cycle between t_1 and t_2 is therefore a function solely of these temperatures, and may be written $f(t_1, t_2)$.

§ 141. **Carnot's function.**

Suppose now that a substance is working in a Carnot cycle between t_1 and t_2, taking in a quantity of heat q_1 from the source, that the external work done in the cycle is $q_1 f(t_1, t_2)$ and that the heat given back to the sink is $q_1\{1 - f(t_1, t_2)\}$. Let another substance work in a

Carnot cycle between the temperatures t_2 and t_3, using the first sink as source and taking from it the quantity of heat $q_1\{1-f(t_1, t_2)\}$ given up by the first substance. It does external work $q_1\{1-f(t_1, t_2)\}f(t_2, t_3)$ and gives up heat $q_1\{1-f(t_1, t_2)\}\{1-f(t_2, t_3)\}$ to a second sink.

Now let another substance work directly from the original source to the second sink, taking in the same quantity of heat q_1 from the source. The work done in the cycle is $q_1 f(t_1, t_3)$ and the heat given back to the sink is $q_1\{1-f(t_1, t_3)\}$.

As in each case the maximum quantity of work has been obtained from the original quantity of heat q_1, the amount of heat given up to the second sink is in each case the same. Therefore

$$q_1\{1-f(t_1, t_2)\}\{1-f(t_2, t_3)\} = q_1\{1-f(t_1, t_3)\}.$$

Suppose that t_1 is constant and t_2, t_3 are varied. Then, since in that case

$$1-f(t_2, t_3) = \frac{1-f(t_1, t_3)}{1-f(t_1, t_2)} = \frac{\mathsf{F}(t_3)}{\mathsf{F}(t_2)},$$

where $\mathsf{F}(t_3)$, $\mathsf{F}(t_2)$ are respectively functions of t_3 and t_2 alone, it must be always possible to write

$$f(t_1, t_2) = 1 - \frac{\mathsf{F}(t_2)}{\mathsf{F}(t_1)}.$$

If q_2 be the heat given up to the sink at temperature t_2, obviously

$$\frac{q_2}{q_1} = \frac{\mathsf{F}(t_2)}{\mathsf{F}(t_1)}.$$

Carnot's function is a quantity μ, such that the efficiency of a reversible engine working between the temperatures t and $t - dt$, where dt is very small, is $\mu\, dt$. If $t_1 - dt$ be put for t_2, $f(t_1, t_2)$ becomes

$$1 - \frac{\mathsf{F}(t_1 - dt)}{\mathsf{F}(t_1)} = 1 - \frac{\mathsf{F}(t_1) - dt\,\mathsf{F}'(t_1)}{\mathsf{F}(t_1)} = \frac{\mathsf{F}'(t_1)}{\mathsf{F}(t_1)}\,dt.$$

Dropping the suffix, we see that μ at temperature t is given by $\mathsf{F}'(t)/\mathsf{F}(t)$.

§ 142. **Kelvin's scale of absolute temperature.**

So far nothing has been said about measurement of temperature. When heat flows from A to B, A is said to be at a higher temperature than B. Temperature is measured by the expansion of an arbitrary substance in terms of an arbitrary scale. The temperature readings of two thermometric substances can be made to agree at any two prearranged temperatures, but then they will in general agree at no other temperature. It is immaterial on what scale t has been measured in the preceding sections, but to fix our ideas we may suppose it to have been the centigrade mercury-in-glass scale.

Kelvin has introduced an absolute scale based on the properties of a perfect heat engine working in a Carnot cycle. This scale is entirely independent of the properties of any thermometric substance.

Let an engine work in a Carnot cycle between the temperatures t_1 and t_2, taking in a quantity of heat q_1 at t_1 and giving out a quantity q_2 at t_2. Then the work done is $q_1 - q_2$. Let a second engine work between t_2 and t_3, taking in q_2 at t_2 and giving out q_3 at t_3, and let the work done by the second engine, $q_2 - q_3$, be equal to $q_1 - q_2$. Let a third engine work between t_3 and t_4, taking in q_3 at t_3 and giving out q_4 at t_4, and let the work done by the third engine, $q_3 - q_4$, be equal to $q_1 - q_2$. And so on.

If q_1, q_2 and t_1 are given, we can in this way arrive at t_4 in three steps and at t_n in $(n-1)$ steps, and the values of t_4 and t_n are absolutely the same, *no matter what the working substance may be.* We take such steps as units on the absolute scale.

From the preceding section,

$$\frac{q_1 - q_2}{q_1} = \frac{\mathsf{F}(t_1) - \mathsf{F}(t_2)}{\mathsf{F}(t_1)}.$$

Since $$q_1 - q_2 = q_2 - q_3 = q_3 - q_4 \text{ etc.},$$

$$\mathsf{F}(t_1) - \mathsf{F}(t_2) = \mathsf{F}(t_2) - \mathsf{F}(t_3) = \mathsf{F}(t_3) - \mathsf{F}(t_4) \text{ etc.}$$

Consequently the increase of the absolute temperature is proportional to the increase of $\mathsf{F}(t)$. We shall denote absolute temperature by T. Assume that when $\mathsf{F}(t)$ is zero, T is zero. This defines the position of zero on the absolute scale and makes T always proportional to $\mathsf{F}(t)$. We have now only to define the size of the unit on the absolute scale. We do that by assuming that there are exactly one hundred of them between 0° and 100° on the centigrade scale.

The efficiency of a substance working in a Carnot cycle between T_1 and T_2 is

$$1 - \frac{\mathsf{F}(t_2)}{\mathsf{F}(t_1)} = 1 - \frac{\mathsf{T}_2}{\mathsf{T}_1}.$$

Let T_1 be constant. Then, as T_2 decreases, the efficiency increases. At zero on the absolute scale, the efficiency is unity, that is, all the heat received from the source is converted into work. The efficiency cannot be greater than unity; T_2 cannot become negative. Hence the absolute zero is the lowest temperature that can be attained by any body. It is not possible to decrease the temperature without limit; there is one definite temperature, the same for all bodies, beyond which it is impossible to go. At this temperature they are entirely devoid of heat. It is found as a result of experiment that the absolute zero is 273° below zero on the centigrade mercury-in-glass scale.

The efficiency of a substance working in a Carnot cycle between T and $\mathsf{T} - d\mathsf{T}$ is

$$1 - \frac{\mathsf{F}(t - dt)}{\mathsf{F}(t)} = 1 - \frac{\mathsf{T} - d\mathsf{T}}{\mathsf{T}} = \frac{d\mathsf{T}}{\mathsf{T}},$$

so that Carnot's function is $1/\mathsf{T}$ when the absolute scale is used.

§ 143. **Entropy.**

Entropy bears somewhat the same relation to the second principle of thermodynamics that energy bears to the first. The name is

due to Clausius. In this section it is defined only for reversible transformations.

If a substance works in a Carnot cycle between the temperatures T_1 and T_2, taking in q_1 at T_1 and giving out q_2 at T_2,

$$\frac{q_1}{T_1} - \frac{q_2}{T_2} = 0.$$

This follows from § 141 and the definition of absolute temperature. If heat imparted to the substance be regarded as positive and heat given out by the body as negative, this equation may be rewritten

$$\frac{q_1}{T_1} + \frac{q_2}{T_2} = 0. \qquad (1)$$

Suppose now that the substance is traversing any reversible cycle not necessarily a Carnot cycle. Then this reversible cycle can be decomposed into a number of elementary Carnot cycles.

For example, let the working substance be a gas and let the closed curve (fig. 70) represent the reversible cycle on the indicator diagram. Draw a number of adiabatic lines so as to divide the area of the curve into elements; let the ends of the elements be bounded by elements of isothermal lines. Then every element of the closed curve is equivalent to an element of an isothermal followed by an element of an adiabatic. The temperatures corresponding to the successive isothermal elements will of course all be different; call them T_1, T_2, T_3, ..., and let q_1, q_2, q_3, ... respectively be the quantities of heat received at the temperatures T_1, T_2, T_3, ... by the substance working round the equivalent stepped curve.

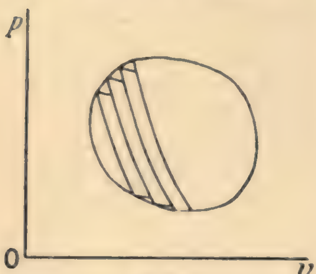

FIG. 70.

Suppose now that we replace the single engine working in the stepped curve by a number of engines, each working in one of the Carnot cycles into which the area is divided, and each completing a cycle in the time taken by the single engine to work round the closed curve. Then, for the system of engines,

$$\frac{q_1}{T_1} + \frac{q_2}{T_2} + \frac{q_3}{T_3} \ldots = 0, \qquad (2)$$

since equation (1) holds for each engine of the system. But the single engine receives q_1, q_2, q_3, ... at temperatures T_1, T_2, T_3, ..., just as the system of engines does and does exactly the same quantity of external work. The single engine and the system of engines are thermodynamically equivalent to one another. Hence equation (2) holds

for the single engine, *i.e.* for an engine working once round a reversible cycle

$$\Sigma \frac{q}{\mathsf{T}} = 0,$$

or, in the limit when the elements are made infinitely small,

$$\int \frac{dq}{\mathsf{T}} = 0.$$

It is not necessary that the working substance should be a gas. The above proof holds for any substance the state of which can be represented on the indicator diagram, *i.e.* which is a function of two independent variables. The theorem is also true when the state of the working substance is a function of more than two independent variables, for the path can still be resolved into elemental isothermals and adiabatics when it can no longer be represented on a plane. Hence

$$\int \frac{dq}{\mathsf{T}} = 0$$

holds for every reversible cyclical process, no matter what the working substance is.

If A and B denote two different states of a substance, which can be connected by a reversible transformation, then $\int \frac{dq}{\mathsf{T}}$ between the limits corresponding to the two states must always have the same value, since the cycle may be completed by a definite invariable transformation.

The change of entropy of the substance in passing from state A to state B is defined by

$$\mathsf{S}_{\mathsf{B}} - \mathsf{S}_{\mathsf{A}} = \int_{\mathsf{A}}^{\mathsf{B}} \frac{dq}{\mathsf{T}}.$$

Since $\int \frac{dq}{\mathsf{T}}$ depends only on the state of the substance, S, the entropy, like U, the intrinsic energy, is a function only of the coordinates defining the state of the body. During any adiabatic transformation dq is always zero, and hence also $\mathsf{S}_{\mathsf{B}} - \mathsf{S}_{\mathsf{A}}$; all adiabatics are therefore isentropics.

§ 144. **Transformation of thermal coefficients.**

Let the state of unit mass of a homogeneous substance be denoted by p, v, t, where p, v, t are connected by the equation

$$f(p, v, t) = 0.$$

This gives
$$\frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial t} dt = 0,$$

and hence
$$\left(\frac{\partial p}{\partial v}\right)_t = -\frac{\dfrac{\partial f}{\partial v}}{\dfrac{\partial f}{\partial p}}, \qquad \text{.................................(3)}$$

where the suffix denotes that during the differentiation t is to be kept constant. From (3),

$$\left(\frac{\partial p}{\partial v}\right)_t\left(\frac{\partial v}{\partial p}\right)_t=1$$

with two similar equations. Also

$$\left(\frac{\partial p}{\partial v}\right)_t\left(\frac{\partial v}{\partial t}\right)_p\left(\frac{\partial t}{\partial p}\right)_v=-\frac{\dfrac{\partial f}{\partial v}\,\dfrac{\partial f}{\partial t}\,\dfrac{\partial f}{\partial p}}{\dfrac{\partial f}{\partial p}\,\dfrac{\partial f}{\partial v}\,\dfrac{\partial f}{\partial t}}=-1.$$

Let a quantity of heat dq be given to the substance. Then, in general, p, v and t will suffer increments dp, dv and dt. Since p, v and t are not independent, dp, dv and dt are not independent; hence dq can be written in either of the two following forms:

$$dq=\gamma_v dt+l_v dv,\quad dq=\gamma_p dt+l_p dp. \qquad\text{.....................(4)}$$

In the above equations l_v is regarded as a function of v and t, and l_p as a function of p and t; l_v is called the latent heat of expansion and l_p the latent heat of pressure variation. They are different from the latent heat of change of state. γ_v and γ_p are respectively the specific heats at constant volume and constant pressure. Of course dq is not a perfect differential.

Equations (4) are quite independent of the laws of thermodynamics.

Similarly, $$dq=\mathsf{M}\,dv+\mathsf{N}\,dp.$$

The six thermal coefficients γ_v, γ_p, l_v, l_p, M, N are not independent. For, eliminating dt from (4),

$$dq=\frac{\gamma_p l_v dv-\gamma_v l_p dp}{\gamma_p-\gamma_v},$$

hence $$\mathsf{M}=\frac{\gamma_p l_v}{\gamma_p-\gamma_v},\quad \mathsf{N}=\frac{\gamma_v l_p}{\gamma_v-\gamma_p}.$$

The first member of (4) can be written

$$dq=\gamma_v dt+l_v\left(\frac{\partial v}{\partial t}\right)_p dt+l_v\left(\frac{\partial v}{\partial p}\right)_t dp,$$

whence $$\gamma_p=\gamma_v+l_v\left(\frac{\partial v}{\partial t}\right)_p,\quad l_p=l_v\left(\frac{\partial v}{\partial p}\right)_t.$$

Similarly, $$\gamma_v=\gamma_p+l_p\left(\frac{\partial p}{\partial t}\right)_v,\quad l_v=l_p\left(\frac{\partial p}{\partial v}\right)_t.$$

Hence four relations exist between the six coefficients, and when two of these are known, the other four can be found.

§ 145. **Carnot's function. Otherwise.**

In deriving Carnot's function in § 141, both the first and second principles of thermodynamics were used. Both the principles of thermodynamics were unknown to Carnot, and he derived the function in the following manner.

Consider the limiting case of Carnot's cycle when the heat taken in becomes infinitesimal. Then ABCD is a parallelogram.

Draw BFP, CEQ parallel to Op. By (4), the heat received from the source is $l_v dv$. The work done in the cycle

$$=\text{ABCD}=\text{BCEF}=\text{BF}\,.\,\text{PQ}.$$

Now $\text{PQ}=dv$ and $\text{BF}=\left(\frac{\partial p}{\partial t}\right)_v dt$, therefore the work done

$$=\left(\frac{\partial p}{\partial t}\right)_v dt\,dv,$$

and the efficiency is $\left(\frac{\partial p}{\partial t}\right)_v \frac{dt}{l_v}=\mu\,dt.$

Therefore $$\mu=\left(\frac{\partial p}{\partial t}\right)_v \Big/ l_v.$$

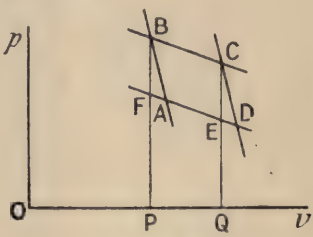

Fig. 71.

By substituting experimental values, Carnot was able to show that μ was independent of the working substance.

§ 146. **The perfect gas.**

We shall now apply some of the results of the preceding sections to the case of a perfect gas.

The heat given to a gas may be used conceivably for three purposes: (1) in external work, (2) in internal work, *i.e.* in moving the molecules further apart against their mutual attractions, and (3) in increasing the kinetic energy of the molecules.

Joule's law (1845) states, that if the temperature does not alter, the intrinsic energy of any mass of gas is constant; it does not depend on the volume or the pressure.

We have already had the equation

$$dq=d\text{U}+d\text{W}.$$

In the case of a gas the only external work done is given by $p\,dv$; hence the equation becomes

$$dq=d\text{U}+p\,dv.$$

In § 144 we had the equation

$$dq=\gamma_v dt+l_v dv.$$

The term $d\text{U}$ includes work done against internal forces as well as increase of kinetic energy of the molecules. In general it depends on v as well as t. In general therefore p is not $=l_v$.

For a gas obeying Joule's law U depends only on t, $d\text{U}=\gamma_v dt$ and $p=l_v$. Joule's law is only approximately fulfilled by actual gases. A perfect gas is defined as one that obeys

(1) Boyle's law, (2) Joule's law.

§ 147. **Clapeyron's formula.**

Consider unit mass of a homogeneous solid or fluid, the only external work done being done against hydrostatic pressure. Then, if t be measured on any arbitrary scale, we have, as in the preceding section,

$$dq = \gamma_v dt + l_v dv, \quad dq = d\mathsf{U} + p\,dv.$$

Hence
$$d\mathsf{U} = \gamma_v dt + (l_v - p)\,dv,$$

and since $d\mathsf{U}$ is a perfect differential,

$$\frac{\partial \gamma_v}{\partial v} = \frac{\partial}{\partial t}(l_v - p) \quad \text{or} \quad \frac{\partial p}{\partial t} = \frac{\partial l_v}{\partial t} - \frac{\partial \gamma_v}{\partial v}. \quad \text{.................(5)}$$

The increase of entropy, $d\mathsf{S}$, is given by

$$d\mathsf{S} = \frac{dq}{\mathsf{T}} = \frac{\gamma_v}{\mathsf{T}}dt + \frac{l_v}{\mathsf{T}}dv,$$

where T is the absolute temperature. Since $d\mathsf{S}$ is a perfect differential,

$$\frac{\partial}{\partial v}\left(\frac{\gamma_v}{\mathsf{T}}\right) = \frac{\partial}{\partial t}\left(\frac{l_v}{\mathsf{T}}\right),$$

i.e.
$$\frac{1}{\mathsf{T}}\frac{\partial \gamma_v}{\partial v} = \frac{1}{\mathsf{T}}\frac{\partial l_v}{\partial t} - \frac{l_v}{\mathsf{T}^2}\frac{\partial \mathsf{T}}{\partial t}, \quad \frac{l_v}{\mathsf{T}}\frac{\partial \mathsf{T}}{\partial t} = \frac{\partial l_v}{\partial t} - \frac{\partial \gamma_v}{\partial v}. \quad \text{.............(6)}$$

In equations (5) and (6) the right-hand sides are the same; hence

$$\frac{l_v}{\mathsf{T}}\frac{\partial \mathsf{T}}{\partial t} = \frac{\partial p}{\partial t} \quad \text{or} \quad l_v = \mathsf{T}\left(\frac{\partial p}{\partial t}\right)_v \frac{\partial t}{\partial \mathsf{T}},$$

which is Clapeyron's formula.

It should be observed, that in deriving Carnot's function in § 145, we have already almost proved this formula by another method.

§ 148. An analogous formula to Clapeyron's formula will now be derived.

$$dq = \gamma_p dt + l_p dp,$$

$$d\mathsf{U} = dq - p\,dv = \gamma_p dt + l_p dp - p\,dv.$$

Now
$$dv = \left(\frac{\partial v}{\partial p}\right)_t dp + \left(\frac{\partial v}{\partial t}\right)_p dt,$$

therefore
$$d\mathsf{U} = \left\{\gamma_p - p\left(\frac{\partial v}{\partial t}\right)_p\right\}dt + \left\{l_p - p\left(\frac{\partial v}{\partial p}\right)_t\right\}dp.$$

Since this is a perfect differential,

$$\frac{\partial}{\partial p}\left\{\gamma_p - p\left(\frac{\partial v}{\partial t}\right)_p\right\} = \frac{\partial}{\partial t}\left\{l_p - p\left(\frac{\partial v}{\partial p}\right)_t\right\},$$

$$\frac{\partial \gamma_p}{\partial p} - \left(\frac{\partial v}{\partial t}\right)_p - p\frac{\partial^2 v}{\partial p\,\partial t_p} = \frac{\partial l_p}{\partial t} - p\frac{\partial^2 v}{\partial t\,\partial p_t} \quad \text{or} \quad \frac{\partial v}{\partial t_p} = \frac{\partial \gamma_p}{\partial p} - \frac{\partial l_p}{\partial t}. \quad \text{.....(7)}$$

Now
$$d\mathsf{S} = \frac{\gamma_p}{\mathsf{T}}dt + \frac{l_p}{\mathsf{T}}dp.$$

Since $d\mathsf{S}$ is a perfect differential,

$$\frac{\partial}{\partial p}\left(\frac{\gamma_p}{\mathsf{T}}\right)=\frac{\partial}{\partial t}\left(\frac{l_p}{\mathsf{T}}\right),$$

$$\frac{1}{\mathsf{T}}\frac{\partial \gamma_p}{\partial p}=\frac{1}{\mathsf{T}}\frac{\partial l_p}{\partial t}-\frac{l_p}{\mathsf{T}^2}\frac{\partial \mathsf{T}}{\partial t} \quad \text{or} \quad -\frac{l_p}{\mathsf{T}}\frac{\partial \mathsf{T}}{\partial t}=\frac{\partial \gamma_p}{\partial p}-\frac{\partial l_p}{\partial t}. \quad \text{.........(8)}$$

Combining (7) and (8), we obtain

$$\frac{\partial v}{\partial t_p}=-\frac{l_p}{\mathsf{T}}\frac{\partial \mathsf{T}}{\partial t} \quad \text{or} \quad l_p=-\mathsf{T}\left(\frac{\partial v}{\partial t}\right)_p\frac{\partial t}{\partial \mathsf{T}},$$

the required formula.

§ 149. If a gas obeys Boyle's law and Joule's law, it must obey Charles' law.

According to Boyle's law, $pv=c$;

hence $p\,dv+v\,dp=0$ or $\left(\frac{\partial p}{\partial v}\right)_t=-\frac{p}{v}$.

If a gas obeys Joule's law, $p=l_v$. Measure the temperature on the absolute scale and apply Clapeyron's formula. Then

$$p=\mathsf{T}\left(\frac{\partial p}{\partial t}\right)_v \quad \text{or} \quad \left(\frac{\partial p}{\partial t}\right)_v=\frac{p}{\mathsf{T}}.$$

By § 144 this is the same as $\left(\frac{\partial t}{\partial p}\right)_v=\frac{\mathsf{T}}{p}$. Now in § 144 we had the formula $\left(\frac{\partial p}{\partial v}\right)_t\left(\frac{\partial v}{\partial t}\right)_p\left(\frac{\partial t}{\partial p}\right)_v=-1$. On substituting for $\left(\frac{\partial p}{\partial v}\right)_t$ and $\left(\frac{\partial t}{\partial p}\right)_v$ the values already found, this becomes

$$-\frac{p}{v}\left(\frac{\partial v}{\partial t}\right)_p\frac{\mathsf{T}}{p}=-1 \quad \text{or} \quad \left(\frac{\partial v}{\partial t}\right)_p=\frac{v}{\mathsf{T}}, \quad \text{..................(9)}$$

which gives $v=v_0\frac{\mathsf{T}}{273}$ when p is constant. This is Charles' law.

From the form of (9) it is obvious that if any two of the three laws—Boyle's, Charles', and Joule's laws—are obeyed, the third must also be obeyed.

§ 150. **Further properties of a perfect gas.**

We can now derive the characteristic equation of a gas. Let p_0 and v_0 be the pressure and volume of unit mass at the temperature 273° on the absolute scale. Let p and v be the pressure and ~~temperature~~ volume of unit mass when the absolute temperature is T.

When the temperature is T let the pressure become p_0, and at the same time let the volume become v'. Then, by Boyle's law,

$$pv=p_0v'.$$

Now let the pressure be kept constant and decrease the temperature to 273°. By Charles' law, v' becomes v_0, v_0 being connected with v' by the equation

$$v' = v_0 \frac{\mathsf{T}}{273}.$$

Hence $$pv = p_0 v_0 \frac{\mathsf{T}}{273} \quad \text{or} \quad pv = \mathsf{RT},$$

where R is constant.

The modulus of elasticity at constant temperature of a gas is the limiting value of the ratio of an increase in pressure to the decrease of volume per unit volume it produces. Let ϵ denote the modulus of elasticity at constant temperature, let δp be the increase of pressure and let δv be the decrease of volume. Then

$$\epsilon = \operatorname*{Lt}_{\delta p = 0} \frac{\delta p}{\frac{\delta v}{v}} = -v\left(\frac{\partial p}{\partial v}\right)_t.$$

The minus sign is necessary since p increases as v decreases. In the case of a perfect gas,

$$\left(\frac{\partial p}{\partial v}\right)_t = -\frac{p}{v}.$$

Hence $\epsilon = p$.

The specific heats at constant pressure and constant volume are respectively γ_p and γ_v. We have

$$d\mathsf{U} = \gamma_v d\mathsf{T},$$

substituting T for t.

$$\gamma_p = \frac{\partial q}{\partial \mathsf{T}_p} = \frac{\gamma_v d\mathsf{T} + p\,dv}{d\mathsf{T}_{p\text{ const.}}} = \gamma_v + p\frac{\partial v}{\partial \mathsf{T}_p}.$$

But since $$pv = \mathsf{RT}, \quad p\frac{\partial v}{\partial \mathsf{T}_p} = \mathsf{R}.$$

Hence $$\gamma_p - \gamma_v = \mathsf{R}.$$

It should be remembered that γ_p and γ_v are here measured in dynamical units.

We shall next find the relation which must hold between p and v during an adiabatic change. During an adiabatic change no heat is received by or escapes from the gas. Hence $dq = 0$ and

$$\gamma_v d\mathsf{T} + p\,dv = 0.$$

But from the equation $$pv = \mathsf{RT},$$

$$d\mathsf{T} = \frac{1}{\mathsf{R}}(p\,dv + v\,dp),$$

whence, on substituting for $d\mathsf{T}$,

$$\gamma_v(p\,dv + v\,dp) + \mathsf{R}p\,dv = 0,$$

$$\text{i.e. } (\gamma_v + \mathsf{R})\frac{dv}{v} + \gamma_v\frac{dp}{p} = 0 \quad \text{or} \quad \gamma_p\frac{dv}{v} + \gamma_v\frac{dp}{p} = 0.$$

On integration this gives $\gamma_p \log v + \gamma_v \log p = \text{const.}$, or if κ be written for γ_p/γ_v, the ratio of the specific heats,

$$\kappa \log v + \log p = \text{const.},$$

$$\log pv^\kappa = \text{const.} \quad \text{or} \quad pv^\kappa = c.$$

This is the required relation. In the integration we have assumed that γ_p is independent of p. According to Regnault this is the case.

If the volume of the gas changes from v_1 to v_2, the external work done is

$$\int_{v_1}^{v_2} p\, dv.$$

If the expansion is an isothermal one, T is independent of v and

$$\int_{v_1}^{v_2} \frac{\mathrm{RT}}{v}\, dv = \mathrm{RT} \log \frac{v_2}{v_1}.$$

If the expansion is an adiabatic one,

$$\int_{v_1}^{v_2} p\, dv = \int_{v_1}^{v_2} \frac{c}{v^\kappa}\, dv = \frac{c}{\kappa - 1}\left(\frac{1}{v_1^{\kappa-1}} - \frac{1}{v_2^{\kappa-1}}\right).$$

§ 151. **Work done by a perfect gas in a Carnot cycle.**

Let the cycle be traversed in the direction ABCD; let p_1v_1, p_2v_2, p_3v_3 and p_4v_4 denote the pressure and volume at A, B, C and D respectively. Let T_1 denote the temperature on AB and T_2 the temperature on CD.

Then, since AB and CD are isothermals and AD and BC adiabatics,

$$p_1v_1 = p_2v_2, \qquad p_3v_3 = p_4v_4, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$p_4v_4^\kappa = p_1v_1^\kappa, \quad p_2v_2 = p_3v_3^\kappa.$$

The work done by the gas during AB is

$$\mathrm{RT}_1 \log \frac{v_2}{v_1}.$$

The work done by the gas during CD is

$$-\mathrm{RT}_2 \log \frac{v_3}{v_4}.$$

The work done during BC is

$$\frac{p_2v_2^\kappa}{\kappa - 1}\left(\frac{1}{v_2^{\kappa-1}} - \frac{1}{v_3^{\kappa-1}}\right),$$

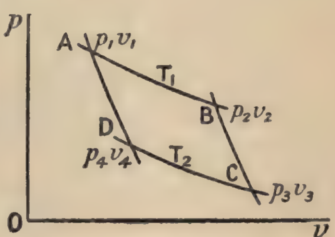

FIG. 72.

$p_2v_2^\kappa$ being substituted for the constant c, and finally the work done during DA is

$$\frac{p_4v_4^\kappa}{\kappa - 1}\left(\frac{1}{v_4^{\kappa-1}} - \frac{1}{v_1^{\kappa-1}}\right).$$

On eliminating p_1, p_2, p_3 and p_4 from equations (10),

$$v_2/v_1 = v_3/v_4.$$

Hence the total work done on the isothermals can be put into the form

$$\mathsf{R}(\mathsf{T}_1 - \mathsf{T}_2)\log\frac{v_2}{v_1}.$$

The total work done on the adiabatics is

$$\frac{1}{\kappa - 1}\left\{p_2v_2^{\kappa}\left(\frac{1}{v_2^{\kappa-1}} - \frac{1}{v_3^{\kappa-1}}\right) + p_4v_4^{\kappa}\left(\frac{1}{v_4^{\kappa-1}} - \frac{1}{v_1^{\kappa-1}}\right)\right\}$$

$$= \frac{1}{\kappa - 1}\left\{p_2v_2 - \frac{p_2v_2^{\kappa}}{v_3^{\kappa-1}} + p_4v_4 - \frac{p_4v_4^{\kappa}}{v_1^{\kappa-1}}\right\}$$

$$= \frac{1}{\kappa - 1}\{p_2v_2 - p_3v_3 + p_4v_4 - p_1v_1\} = 0.$$

§ 152. **Entropy of a perfect gas.**

Given the entropy, S_0, of unit mass of a perfect gas when its pressure and volume are p_0, v_0, it is required to find the entropy, S, when its pressure and volume are p and v.

$$d\mathsf{S} = \frac{dq}{\mathsf{T}} = \frac{\gamma_v d\mathsf{T} + p\,dv}{\mathsf{T}}.$$

Since $pv = \mathsf{RT}$, $\log p + \log v = \log \mathsf{R} + \log \mathsf{T}$.

Hence $$\frac{dp}{p} + \frac{dv}{v} = \frac{d\mathsf{T}}{\mathsf{T}}.$$

Substitute for $\frac{d\mathsf{T}}{\mathsf{T}}$ in the expression for $d\mathsf{S}$; then, since the entropy depends only on the state of the body, the right hand must be integrable.

On substitution,

$$d\mathsf{S} = \gamma_v\left(\frac{dp}{p} + \frac{dv}{v}\right) + \mathsf{R}\frac{dv}{v} = \gamma_v\frac{dp}{p} + \gamma_p\frac{dv}{v}.$$

Hence $$\mathsf{S} = \gamma_v \log p + \gamma_p \log v + \mathsf{C},$$

where C is the constant of integration. When C is determined by substituting the initial conditions,

$$\mathsf{S} - \mathsf{S}_0 = \gamma_v \log\frac{p}{p_0} + \gamma_p \log\frac{v}{v_0}.$$

EXAMPLES.

1. Show that, for a perfect gas undergoing an adiabatic transformation,

$$\frac{v_1^{\kappa-1}}{\mathsf{T}_2} = \frac{v_2^{\kappa-1}}{\mathsf{T}_1}.$$

2. Supposing the earth to have been originally a nebulous mass dissipated through space, show that the heat produced by its condensation is 90 times the amount required to raise an equal mass of water from 0° to 100° C.

3. If the sun be contracting in consequence of its own attraction, show that an annual contraction of $7{\cdot}7\ 10^{-8}$ of a diameter is sufficient to maintain its temperature constant. (The heat emitted by the sun in one year would raise the temperature of an equal mass of water 2° C.)

4. A gas at p_1 and v_1 is allowed to expand into a perfectly empty vessel and its pressure and volume become p_2 and v_2. No heat is imparted to or taken from it during the process. Determine the change in temperature and show that the increase of entropy is

$$\mathsf{R}\log\frac{v_2}{v_1}.$$

5. A vertical cylinder of cross-sectional area A is filled with gas at the atmospheric pressure p_1, the absolute temperature being T_1, and closed by a piston on which is placed a weight w, which pushes it down. No heat passes in or out of the cylinder. Determine the temperature and increase of entropy of unit mass when equilibrium is established. (Note that the change is irreversible and not adiabatic.)

6. Compressed air is contained inside a vessel at a pressure p_1, a little greater than the atmospheric pressure, the temperature of the air being the same as the temperature of the atmosphere. The vessel is opened for a moment to allow the pressure to change adiabatically to that of the atmosphere and then closed. The pressure then rises to p_2 as the temperature of the air attains its old value. Show that the ratio of the specific heats of air is given by

$$\kappa=\frac{\log p_0-\log p_1}{\log p_2-\log p_1},$$

p_0 being the atmospheric pressure.

7. Calculate the difference between the two specific heats of air, being given that a cubic metre of air at a temperature of 0° C. and under a pressure of 76 cms. of mercury, the density of mercury being 13·6, weighs 1·2932 kilogrammes. State any assumptions made in the calculation.

8. The state of unit mass of a perfect gas is represented on a coordinate diagram by its entropy and its absolute temperature. Show that if the gas is put through a reversible cycle, the area of the closed curve traced out is equal to the area of the corresponding curve on the indicator diagram.

9. For any gas whose specific gravity referred to air is ρ, show that

$$\gamma_p=\gamma_v+\frac{0{\cdot}069}{\rho}.$$

10. Show that the quantity of heat which must be imparted to a gas to enable it to expand at a constant pressure p_1 from the volume v_1 to the volume v_2 is

$$\frac{\gamma_p}{\mathsf{R}}p_1(v_2-v_1).$$

11. Calculate, in dynamical units, the increase of entropy of a kilogramme of water which is raised in temperature from 0° C. to 100° C. and evaporated at the latter temperature.

12. A body is surrounded by a medium of unalterable temperature T_0 and is cooled to that temperature by the performance of work by a perfect engine at the expense of its heat. Prove that the whole work done is

$$\mathsf{U}-\mathsf{U}_0-\mathsf{T}_0(\mathsf{S}-\mathsf{S}_0),$$

where U, U_0 denote the internal energies, S, S_0 the entropies, in the initial and final states respectively.

13. A system of any bodies isolated from without is imagined divided up into parts, the thermal capacity of each of which is the same. Show that the utmost useful work obtainable from the system, by perfect engines working between its parts, is equal to the product of the thermal capacity of the whole system by the excess of the arithmetical mean of the temperatures of the parts of the system over their geometrical mean.

14. If two bodies of equal thermal capacity at absolute temperatures T_1, T_2 respectively are brought to the same temperature by a reversible process, their final temperature is $\sqrt{T_1 T_2}$.

A rod of length l coated with an opaque substance is heated so that the temperature at a distance x from an end is $a+bx$. The temperatures of the different parts of the rod are then allowed to become equal by conduction ; find the energy dissipated.

15. A body is put through a reversible cycle of operations consisting of two opposite isothermal strains of a given type and (equal and opposite) small changes of temperature at constant strain. Show by the consideration of the cycle of operations, that if W be the work done by external forces in the first isothermal strain, the change of the body's internal energy arising from that strain is

$$W - T\frac{\partial W}{\partial T},$$

where T denotes absolute thermodynamic temperature.

16. Show that if γ_p, γ_v are the specific heats of a body at constant pressure and volume respectively, at absolute temperature T,

$$\frac{\partial \gamma_v}{\partial v} = T\frac{\partial^2 p}{\partial T^2} \quad \text{and} \quad \frac{\partial \gamma_p}{\partial p} = -T\frac{\partial^2 v}{\partial T^2}.$$

It is stated that the specific heat of carbon dioxide at constant pressure increases with the pressure, attaining a maximum at about 110 atmospheres, after which it diminishes. How would you expect the coefficient of expansion to alter with temperature ?

17. Assuming Nernst's theorem, that the entropy of all solids is zero at the absolute zero, show that the specific heat of every solid becomes infinitely small as its temperature approaches the absolute zero.

§ 153. **The porous plug experiment.**

So far the gases considered have been perfect gases. Real gases do not obey Boyle's law absolutely. It was shown in 1854 by Kelvin and Joule in a classic experiment that Joule's law is also only approximately fulfilled.

In this experiment a steady stream of gas passed through a copper spiral in a water bath and then through a tube in which there was a plug of cotton wool. The pressure of the gas fell considerably in passing through the plug. The temperature of the gas was read by two sensitive thermometers before and after its passage through the plug. The tube containing the plug was surrounded with water in order to keep its temperature steady.

Immediately on passing through the plug the temperature of the gas suffered a small change θ indicated by the difference of the readings on the two thermometers (any arbitrary scale, positive for an increase).

But when heat had time to flow in or out, this change disappeared and the temperature recovered its former value. Let p_1, v_1 be the pressure and volume of unit mass before passing through the plug, and p_2, v_2 the pressure and volume of unit mass after passing through the plug, when equilibrium is re-established. Then, since it is the volume and not the pressure that alters while the temperature recovers its original value, the heat escaping from unit mass during this process, measured in dynamical units, is

$$\gamma_p \theta.$$

Let us now find the work done by external forces on the unit mass in its passage through the plug. In fig. 73 let the plug be represented by the aperture a. Let σ be the cross-sectional area of the tube. Consider the gas between the two planes A and A′. After a short time it will be enclosed between the planes B and B′; $\mathsf{A'B'} > \mathsf{AB}$ since the pressure is less to the right of the plug. During the time considered the external work done on the mass of gas considered is $p_1\sigma\mathsf{AB}$; the external work done by the mass of gas considered is $p_2\sigma\mathsf{A'B'}$ and the quantity of gas that passes through the aperture is $\sigma\mathsf{AB}/v_1 = \sigma\mathsf{A'B'}/v_2$. Hence, when unit mass passes through the plug, the total external work done on it is

$$p_1v_1 - p_2v_2.$$

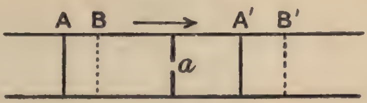

FIG. 73.

This gives the external work done on it from plane A to plane A′, from the plane where the pressure and volume of unit mass are p_1, v_1 to the plane where they are p_2, v_2.

Let $d\mathsf{U}$ be the increase between the two planes in the intrinsic energy of unit mass. Then

$$d\mathsf{U} = -\gamma_p\theta + (p_1v_1 - p_2v_2). \quad \text{.......................(11)}$$

The work done against friction in the plug does not enter into this equation, because it returns to the gas immediately in the form of heat.

If the gas obeys Boyle's law, $p_1v_1 - p_2v_2 = 0$, since the temperature is the same at A and A′. Also, if it obeys Joule's law, $d\mathsf{U}$ equals zero. Hence, for a perfect gas, θ must be zero. If $d\mathsf{U}$ is positive, it of course represents increase of intrinsic energy due to work done against internal forces; it is a gain of molecular potential energy.

The advantage of the arrangement employed by Kelvin and Joule is, that owing to the small thermal capacity of gas θ is relatively large and is easier to detect than if the work done against internal forces were measured by the change in temperature of a mass of water. Also, owing to a stream of gas being employed instead of a definite and necessarily smaller quantity, any disturbing action due to the sides is minimised.

For air and carbon dioxide $\gamma_p\theta$ was found to be negative and $p_1v_1 - p_2v_2$ was also negative but not at all so large; for hydrogen θ was positive but extremely small and $p_1v_1 - p_2v_2$ positive. For all three $d\mathsf{U}$ is positive.

Let us return to equation (11). Write dq for $-\gamma_p\theta$, the heat received per unit mass of gas. If the complete change had been a reversible one, which it is not, we would have had

$$dq = \gamma_p dt + l_p dp \quad \text{or} \quad dq = l_p dp,$$

since here $dt = 0$. The increase in intrinsic energy would then have been given by

$$d\mathsf{U} = l_p dp - p\,dv.$$

But the increase in intrinsic energy depends only on the state of the gas and not on the manner in which that state is brought about. Hence the above expression for $d\mathsf{U}$ may be substituted in (11). Also, since the deviation from Boyle's law is small, $p_2v_2 - p_1v_1$ may be written $d(pv)$. Equation (11) thus becomes

$$dq = l_p dp - p\,dv + d(pv) = (l_p + v)\,dp$$

or

$$\frac{\partial q}{\partial p} = l_p + v. \qquad \text{(12)}$$

Here dp is the increase of pressure in passing through the plug. We have, from § 148, putting $t = \mathsf{T}$,

$$l_p = -\mathsf{T}\frac{\partial v}{\partial \mathsf{T}_p} \quad \text{or} \quad l_p = -\frac{\partial v}{\partial (\log \mathsf{T})_p}.$$

On substituting, (12) becomes

$$\frac{\partial v}{\partial (\log \mathsf{T})_p} = v - \frac{\partial q}{\partial p}. \qquad \text{(13)}$$

By the porous plug experiment $\frac{\partial q}{\partial p}$ can be found as a function of v, and hence the absolute temperature can be found as a function of the volume.

Rose-Innes (1901) finds that the largest correction necessitated by the $\frac{\partial q}{\partial p}$ term on the scale of the nitrogen thermometer between 0° and 100° C. is $-0{\cdot}0026°$, and that the largest correction on the scale of the hydrogen thermometer within the same range is only $-0{\cdot}0007°$.

We shall now derive the characteristic equation of a real gas on the basis of the porous plug experiment.

It is found experimentally that $\frac{\partial q}{\partial p}$ is nearly proportional to $\frac{1}{\mathsf{T}^2}$. Hence, on writing $\frac{\partial q}{\partial p} = \frac{n}{\mathsf{T}^2}$, (13) becomes

$$-\frac{n}{\mathsf{T}^2} = \mathsf{T}\frac{\partial v}{\partial \mathsf{T}_p} - v.$$

On multiplying both sides by $\frac{d\mathrm{T}}{\mathrm{T}^2}$, this becomes

$$-\frac{nd\mathrm{T}}{\mathrm{T}^4}=d\left(\frac{v}{\mathrm{T}}\right),$$

and, on integrating,

$$\frac{v}{\mathrm{T}}=\frac{n}{3\mathrm{T}^3}+\mathrm{H}, \qquad (14)$$

where H is the constant of integration. Assume that, when T is very great, the gas has the properties of an ideal gas, for which $pv=\mathrm{RT}$. Then $\mathrm{H}=\frac{\mathrm{R}}{p}$; hence (14) becomes

$$pv=\mathrm{RT}+\frac{np}{3\mathrm{T}^2}. \qquad (15)$$

If n be put $=0$, this reduces to the ordinary form of the characteristic equation.

§ 154. **Van der Waals' equation.**

Equation (15) is intended to hold only for a gas. Van der Waals has proposed the following characteristic equation,

$$\left(p+\frac{a}{v^2}\right)(v-b)=\mathrm{RT}, \qquad (16)$$

as applicable both to the liquid and gaseous states. When p and v are large, it is obvious that this equation reduces to the ordinary form for a perfect gas. Van der Waals arrived at the additional terms on the basis of the kinetic theory of gases. We shall not go into the method of deriving them here, but merely remark that they look plausible; when T is zero, the volume does not vanish, but tends towards a fixed value b, and when the volume is small, the additional term a/v^2 diminishing the pressure becomes appreciable, and may be supposed due to the attraction between the molecules.

On rewriting (16), it becomes

$$p=-\frac{a}{v^2}+\frac{\mathrm{RT}}{v-b}. \qquad (17)$$

If T be regarded as constant, this is the equation to an isothermal. For a given value of p the equation is a cubic in v. Now a cubic equation must have either one or three real roots. Hence, for T constant and a given value of p, the equation gives either one or three values for the volume of unit mass. We have the one real value in the case of a gas; then T is greater than the critical temperature. We have the three real values in the case when T is less than the critical temperature; the greatest of these values gives the volume of the saturated vapour, the least gives the volume of the liquid into which it condenses, while the middle value has no practical significance.

At the critical temperature all three values of v are equal, k say. Writing Van der Waals' equation at length,

$$v^3 - v^2\left(b + \frac{\mathsf{RT}}{p}\right) + v\frac{a}{p} - \frac{ab}{p} = 0,$$

and comparing it with the equation

$$(v-k)^3 = 0,$$

we see that $$3k = b + \frac{\mathsf{RT}}{p}, \quad 3k^2 = \frac{a}{p} \quad \text{and} \quad k^3 = \frac{ab}{p}.$$

From the second and third of these equations we find that $k = 3b$, and on substituting this value we find altogether for the critical point,

$$v = 3b, \quad p = \frac{a}{27b^2} \quad \text{and} \quad \mathsf{T} = \frac{8a}{27b\mathsf{R}}. \qquad \text{(18)}$$

§ 155. **Effect of pressure on the freezing point.**

We had $$dq = \gamma_v d\mathsf{T} + l_v dv.$$

If this equation is applied to fusion or vaporisation, it becomes simply

$$dq = l_v dv,$$

since during these changes the temperature remains constant. If the temperature is measured on the absolute scale, Clapeyron's formula is

$$l_v = \mathsf{T}\frac{\partial p}{\partial \mathsf{T}_v}.$$

Eliminating l_v, and at the same time writing $dv = \mathsf{V} - v$, where v and V are respectively the volumes of unit mass before and after the change of state, we obtain

$$dq = \mathsf{T}\frac{\partial p}{\partial \mathsf{T}_v}(\mathsf{V} - v).$$

Let the equation be applied to vaporisation and let L be the latent heat of vaporisation. Then

$$\mathsf{L} = \mathsf{T}\frac{\partial p}{\partial \mathsf{T}_v}(\mathsf{V} - v), \qquad \text{(19)}$$

where v is the volume of unit mass of water, V the volume of unit mass of saturated steam and T the temperature of change of state.

Hence, if $\mathsf{V} > v$, *i.e.* if the substance expands on vaporising, $\frac{\partial p}{\partial \mathsf{T}_v}$ is positive and increase of pressure raises the boiling point.

If the equation be applied to the melting of ice, $\mathsf{V} - v$ is negative, since the volume of unit mass of ice is greater than the volume of unit mass of water. Hence $\frac{\partial p}{\partial \mathsf{T}_v}$ is negative; an increase in pressure lowers the melting point of ice. This result was deduced by Prof. James Thomson in 1850 and was subsequently verified by Lord Kelvin. The amount of the change can readily be calculated from the formula. For,

putting $T=273$, $L=80$ cals. $=336\ 10^7$ ergs, $V=1\cdot 00$ c.cs., $v=1\cdot 09$ c.cs., $dp=1$ atmosphere $=10^6$ dynes/sq. cm.,

$$dT = \frac{-273\ 10^6\ \cdot 09}{336\ 10^7} = -0\cdot 0073^\circ \text{ C.}$$

Equation (19) can be used for calculating the density of saturated steam, which is difficult to determine experimentally. For $\frac{\partial p}{\partial T_v}$ is known from the relation between the pressure and temperature of saturated steam, v can be taken as 1 and the value of L at different temperatures can be found from Regnault's formula for the total heat of steam, $Q=606\cdot 5-0\cdot 305t$. The total heat of steam at any temperature t is the quantity of heat necessary to raise 1 gramme of water from 0° to t° C. and to evaporate it at that temperature.

Equation (19) can be derived very easily from consideration of a Carnot cycle. Suppose we have some water and aqueous vapour all at a uniform temperature T contained by a cylinder and piston, and that the piston is raised so that exactly 1 gramme of water vaporises, the temperature being kept constant all the time. The isothermal traversed is represented on the indicator diagram by the straight line AB. Next let the water and vapour expand adiabatically. The temperature will fall to $T-dT$. In reality some aqueous vapour also condenses, but we neglect that. The adiabatic is BC. Next compress the working substance isothermally so that exactly 1 gramme condenses. The state of the substance is now represented in the indicator diagram by the point D. Finally compress adiabatically so that the temperature rises once more to T.

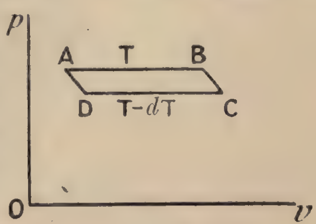

FIG. 74.

$AB=V-v$. The perpendicular distance between AB and DC is $\frac{\partial p}{\partial T_v}dT$. Hence the whole work done in the cycle, *i.e.* the area of the parallelogram, is given by

$$(V-v)\frac{\partial p}{\partial T_v}dT. \qquad (20)$$

The heat taken in is L. The efficiency of the cycle is $\frac{dT}{T}$; hence the work done is

$$L\frac{dT}{T}. \qquad (21)$$

On equating (20) and (21) we get (19) again.

§ 156. **The specific heat of saturated vapour.**

The specific heat of a saturated vapour (γ') is the quantity of heat necessary to raise the temperature of unit mass 1° C., keeping it saturated.

Consider a mixture of liquid and saturated vapour, the total mass of the mixture being 1 gramme and the mass of the vapour being x grammes.

Let the temperature be T and let the latent heat of vaporisation be L. Then the system is completely defined by the two independent coordinates x and T. Let γ be the specific heat of the liquid.

Let a quantity of heat dq be given to the system. Then

$$dq = \{(1-x)\gamma + x\gamma'\}\,d\mathsf{T} + \mathsf{L}\,dx, \quad d\mathsf{S} = \frac{(1-x)\gamma + x\gamma'}{\mathsf{T}}\,d\mathsf{T} + \frac{\mathsf{L}}{\mathsf{T}}\,dx.$$

Since this is a perfect differential,

$$\frac{\gamma'-\gamma}{\mathsf{T}} = \frac{1}{\mathsf{T}}\frac{\partial \mathsf{L}}{\partial \mathsf{T}} - \frac{\mathsf{L}}{\mathsf{T}^2} \quad \text{or} \quad \gamma' - \gamma = \frac{\partial \mathsf{L}}{\partial \mathsf{T}} - \frac{\mathsf{L}}{\mathsf{T}}.$$

Everything except γ' can be easily determined, and hence the latter can be calculated.

For water γ' is negative, *i.e.* if the temperature of aqueous vapour be raised, heat must be taken from it in order to keep it saturated. For ether γ' is positive.

§ 157. **Change of temperature produced in a wire by stretching it.**

Consider a wire hanging vertically with its upper end fixed and a pan attached to its lower end for the purpose of holding weights. Then the state of the wire at any time may be regarded as a function of two independent variables, F the stretching force and T the absolute temperature. Let x denote the vertical displacement of the lower end of the wire.

If T and F suffer small changes, the heat received by the wire is given by

$$dq = \gamma\,d\mathsf{T} + a\,d\mathsf{F}. \qquad (22)$$

In this equation γ is not the specific heat, but the thermal capacity of the whole wire.

The work done on the wire when x is increased is $\mathsf{F}\,dx$. Let U be the intrinsic energy of the wire and S its entropy. Then

$$d\mathsf{U} = dq + \mathsf{F}\,dx = \gamma\,d\mathsf{T} + a\,d\mathsf{F} + \mathsf{F}\frac{\partial x}{\partial \mathsf{T}}\,d\mathsf{T} + \mathsf{F}\frac{\partial x}{\partial \mathsf{F}}\,d\mathsf{F}$$

$$= \left(\gamma + \mathsf{F}\frac{\partial x}{\partial \mathsf{T}}\right)d\mathsf{T} + \left(a + \mathsf{F}\frac{\partial x}{\partial \mathsf{F}}\right)d\mathsf{F}$$

and

$$d\mathsf{S} = \frac{\gamma}{\mathsf{T}}\,d\mathsf{T} + \frac{a}{\mathsf{T}}\,d\mathsf{F}.$$

Since these are perfect differentials, we have the following two relations:

$$\frac{\partial \gamma}{\partial \mathsf{F}} + \frac{\partial x}{\partial \mathsf{T}} = \frac{\partial a}{\partial \mathsf{T}}$$

and

$$\frac{1}{\mathsf{T}}\frac{\partial \gamma}{\partial \mathsf{F}} = \frac{1}{\mathsf{T}}\frac{\partial a}{\partial \mathsf{T}} - \frac{a}{\mathsf{T}^2} \quad \text{or} \quad \frac{\partial \gamma}{\partial \mathsf{F}} = \frac{\partial a}{\partial \mathsf{T}} - \frac{a}{\mathsf{T}}.$$

Eliminating $\frac{\partial \gamma}{\partial \mathsf{F}} - \frac{\partial a}{\partial \mathsf{T}}$ between the two equations, we obtain

$$\frac{\partial x}{\partial \mathsf{T}} = \frac{a}{\mathsf{T}}. \quad\text{.................................}(23)$$

Since the wire expands on heating, $\frac{\partial x}{\partial \mathsf{T}}$ is positive, and hence a is positive. If the wire is suddenly stretched, the change may be regarded as an adiabatic one, at least approximately. Putting $dq = 0$ in (22), we obtain

$$\gamma\, d\mathsf{T} + a\, d\mathsf{F} = 0.$$

Now γ, a and $d\mathsf{F}$ are all positive; $d\mathsf{T}$ must be negative. Hence, on stretching the wire, its temperature falls.

A very accurate experimental investigation of this fall in temperature has been made, and its magnitude has been found to agree with the formula within the error of observation.

India-rubber contracts when it is heated. Consequently for it a is negative, and, on its being stretched, its temperature rises.

Equation (23) can also be derived by considering a cycle. For, let the wire expand isothermally from A to B at temperature T taking in

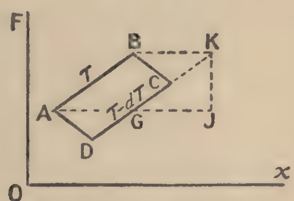

Fig. 75.

a quantity of heat dq, let it expand from B to C adiabatically, let it contract isothermally from C to D at $\mathsf{T} - d\mathsf{T}$ giving up heat and finally let it contract adiabatically from D to A. The efficiency of the cycle is $\frac{d\mathsf{T}}{\mathsf{T}}$ and $dq = a\, d\mathsf{F} = a\overline{\mathsf{JK}}$. Hence the work done in the cycle is

$$\frac{d\mathsf{T}}{\mathsf{T}}\, dq = \frac{a\, d\mathsf{T}\, \overline{\mathsf{JK}}}{\mathsf{T}}.$$

This is equal to the area of

$$\mathsf{ABCD} = \mathsf{KGAB} = \mathsf{KJ} \times \mathsf{BK} = \mathsf{KJ}\frac{\partial x}{\partial \mathsf{T}}\, d\mathsf{T}.$$

On equating, we obtain

$$\overline{\mathsf{KJ}}\frac{\partial x}{\partial \mathsf{T}}\, d\mathsf{T} = \frac{a\, d\mathsf{T}\, \overline{\mathsf{KJ}}}{\mathsf{T}},$$

whence
$$\frac{\partial x}{\partial \mathsf{T}} = \frac{a}{\mathsf{T}}.$$

§ 158. **Effect of temperature on the E.M.F. of a reversible cell.**

By a reversible cell is meant one like a storage battery, in which all the chemical changes are gone through in the reverse order, when the current runs the other way. Let T be the absolute temperature of the cell and e the quantity of electricity which has passed through it. Then, according to the laws of electrolysis, these two quantities completely define its state. When a quantity e passes through the cell, let he be the chemical energy liberated and λe the heat absorbed by the cell.

The cell is to be the working substance. Suppose that it is used to drive a motor and that the motor works without friction. Let R be the total resistance of the circuit and E the E.M.F. of the cell; then the rate at which energy is being dissipated in heat in the circuit is $\mathsf{R}i^2$, and the rate at which useful work is being done is $\mathsf{E}i - \mathsf{R}i^2$, i being the current in the circuit. The ratio of these quantities,

$$\frac{\mathsf{R}i^2}{\mathsf{E}i - \mathsf{R}i^2} \quad \text{or} \quad \frac{\mathsf{R}i}{\mathsf{E} - \mathsf{R}i},$$

approaches zero when i is made very small. If i is very small, all the energy is thus spent in useful work, and the transformation taking place in the cell may be regarded as a reversible one. We shall suppose i to be very small.

When the temperature of the cell is T, let a quantity of electricity e pass through it. The heat absorbed is then λe and the work done by the cell is $\mathsf{E}e$. Next break the circuit, cool the cell adiabatically to $\mathsf{T} - d\mathsf{T}$ and let its E.M.F. become $\mathsf{E} - d\mathsf{E}$. Then make the circuit and work the motor backwards so as to charge the cell isothermally, letting a quantity e pass through it. The work done on the cell is in this case $(\mathsf{E} - d\mathsf{E})e$. Finally, break the circuit and suppose that chemical changes take place in the cell so that it heats adiabatically again to T.

The efficiency of the cycle is $\dfrac{d\mathsf{T}}{\mathsf{T}}$, the heat taken in is λe, and hence the work done is $\lambda e \dfrac{d\mathsf{T}}{\mathsf{T}}$. But the total work done by the cell in the cycle is

$$\mathsf{E}e - (\mathsf{E} - d\mathsf{E})e = e\, d\mathsf{E}.$$

Hence $$\lambda e \frac{d\mathsf{T}}{\mathsf{T}} = e\, d\mathsf{E} \quad \text{or} \quad \lambda = \mathsf{T}\frac{\partial \mathsf{E}}{\partial \mathsf{T}}. \quad \text{.....................(24)}$$

But, when the temperature of the cell remains constant, $\mathsf{E}e$ is equal to the heat taken in plus the chemical energy liberated, that is,

$$\mathsf{E}e = \lambda e + he.$$

Thus, on substituting for λ in equation (24), we obtain the final result

$$\mathsf{E} = h + \mathsf{T}\frac{\partial \mathsf{E}}{\partial \mathsf{T}}.$$

Generally the E.M.F. increases with the temperature; E is then greater than h, *i.e.* heat is absorbed by the cell, and the latter will cool if no heat is supplied to it.

§ 159. Second definition of entropy.

Suppose that a substance is working in a Carnot cycle between temperatures T_1 and T_0, taking in a quantity of heat q_1 at T_1 and giving out q_0 at T_0, q_1 and q_0 being measured in dynamical units. Then the work done in the cycle is $q_1 - q_0$. Let T_0 be the lowest temperature available for a sink. Then q_0 is heat energy that cannot be turned into work, or, in other words, it is unavailable energy. Now

$$q_0 = \frac{T_0 q_1}{T_1}.$$

The wholly unavailable energy associated with a given quantity of heat is thus:

(1) directly proportional to the lowest absolute temperature available for a sink and

(2) inversely proportional to the temperature of the body which the heat is leaving.

We may regard $\frac{q_1}{T_1}$ as a measure of unavailability or factor which only requires to be multiplied by any assumed auxiliary temperature T_0 in order to give the quantity of unavailable energy relative to that temperature.

If from any cause whatever the unavailable energy of a body with reference to an auxiliary medium of temperature T_0 undergoes any (positive or negative) increase and if this increase be divided by the temperature T_0, the quotient is called the increase of entropy of the body.

This definition is much more suitable than the former one for irreversible phenomena. It is easy to see that the two definitions are identical for the case of a perfect gas and a reversible cycle. Both definitions define only increase of entropy, and hence involve an arbitrary constant.

If a system is taken from a state A to a state B by a reversible transformation,

$$S_B - S_A = \Sigma \int_A^B \frac{dq}{T}.$$

If the transformation is irreversible,

$$S_B - S_A > \Sigma \int_A^B \frac{dq}{T},$$

because irreversible changes, for example friction, loss of heat by diffusion, etc., always involve an increase of unavailable energy. The summation is to be taken over the different bodies of the system, and S here denotes the entropy of the whole system.

§ 160. **The second principle of thermodynamics.**

In § 134 we gave the statement of the second principle due to Clausius. Lord Kelvin has stated it in the following manner:—It is impossible by means of inanimate material agency to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects; and Clerk Maxwell has given it another enunciation, namely:—It is impossible, by the unaided action of natural processes, to transform any part of the heat of a body into mechanical work, except by allowing heat to pass from that body into another at a lower temperature. Perhaps the following enunciation is the most useful of all:—The entropy of an isolated system of bodies cannot decrease; it remains constant in the reversible processes and increases in the irreversible processes that take place within the system. As a result of this definition, the second principle of thermodynamics is sometimes referred to as the principle of the increase of entropy. It is connected with the doctrine of the dissipation or degradation of energy.

The second principle of thermodynamics thus states the direction in which changes in nature are taking place. There is in nature a quantity which changes always in the same sense in all natural processes.

If an irreversible change can take place, it will. The reversible cycles which we have studied are either cases of equilibrium or limiting cases of irreversible cycles. Irreversible changes, on account of their complexity, do not lend themselves readily to calculation or illustration.

In the second principle of thermodynamics we have a glimpse of a very general law, that possibly transcends physics and which is not yet fully understood. Hence the different ways of stating it.

EXAMPLES.

1. Prove that if T denote absolute temperature, $d\mathsf{T}$ the heating effect due to the flow through the porous plug in Kelvin and Joule's experiment from pressure p to pressure $p+dp$, v the volume of a gramme of the gas, γ_p, γ_v the specific heats at constant pressure and constant volume respectively,

$$\frac{1}{\mathsf{T}}\frac{\partial \mathsf{T}}{\partial v_p}=\frac{1+\mu\dfrac{\partial p}{\partial \mathsf{T}_v}}{v-\mu\gamma_v}=\frac{1}{v-\mu\gamma_p},$$

where μ denotes $\dfrac{d\mathsf{T}}{dp}$.

Hence, assuming the equation $pv=\mathsf{RT}$ for the gas, show that

$$\gamma_p-\gamma_v=p\left(1-\frac{\mu}{v}\gamma_p\right)\frac{\partial v}{\partial \mathsf{T}_p}.$$

Compare this with the result of the ordinary supposition as to the difference between the specific heats.

2. Assuming Van der Waals' equation, show that if v_1, v_2 are the specific volumes of a liquid and its vapour in contact at temperature T, the latent heat of vaporisation is

$$\mathsf{RT}\log_e\frac{v_2-b}{v_1-b}.$$

3. Show that if γ, γ_1 are the thermal capacities of a stretched wire under constant tension and at constant length respectively, l the length of the wire at absolute temperature T under stretching force F and α the coefficient of linear expansion, then

$$\gamma_1=\gamma+\alpha l\mathsf{T}\left(\frac{\partial\mathsf{F}}{\partial\mathsf{T}}\right)_l.$$

4. If F denote the superficial tension of a film of water, experiment has shown that $\frac{\partial\mathsf{F}}{\partial\mathsf{T}}=-0{\cdot}1425$ in dynes per cm. per degree Centigrade. Hence show that about half as much energy must be given to the film in an isothermal extension to prevent its temperature from sinking as is spent by external forces in stretching the film.

5. A wire is suspended vertically, the upper end being fixed and the lower end being stretched by a force F and twisted by a couple L. If x denote the extension and θ the angle of twist of the lower end of the wire, show that

$$\frac{\partial x}{\partial\mathsf{L}}=\frac{\partial\theta}{\partial\mathsf{F}}.$$

INDEX.

The references are to pages.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.